# Sequential Screening with Personal Selling †

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**Abstract**

I analyze a monopolistic screening model, where the buyer's type is initially unknown to both market sides. The seller engages in costless sequential communication with the buyer before presenting a final product offer. At each communication period, the seller selects a threshold and discloses to the buyer whether his type is above or below it. The optimal strategy for the seller is to gradually disclose information about the buyer's type, starting from the bottom. Compared to the standard monopolistic screening, this approach enables the seller to extract the entire surplus not only from the lowest served type but from a whole range of lower types. I also introduce an analog of a virtual type for a learning-buyer environment and examine the consequences of the buyer's limited knowledge for consumer welfare.

†This paper was anticipated by Heumann (2020).

*Keywords*: Sequential Screening, Monopoly Pricing, Seller-Buyer Communication, Learning Buyer, Salesperson.

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*It is a great advantage to show him the lower-priced registers, leading him from one to another up to the better ones, ..., yet all those shown previously to the one you believe he ought to have should be treated as if they were a single flight of minor steps leading up to the one important landing.*

— *National Cash Register Manual*, as quoted in Russell (1912)

# 1 Introduction

In many market interactions, buyers may have limited ability to understand their own preferences. Certain product types are particularly hard for buyer evaluation. A prominent example is "look and feel" goods (e.g. artwork, used cars, tailored suits) that consumers can only evaluate upon direct interaction (De Figueiredo (2000)). Given that buyers' learning about such products takes its place at the store, we could expect that it can be subjected to the seller's influence. A case in point is Olshavsky (1973) who finds that the salespeople often manipulate buyer's in-store experience and, in particular, their evaluation of available products: "...in most cases, it is the salesman and not the customer who determines the extent of search and evaluation of alternatives ...[and] the salesman typically selects the order and the number of alternatives evaluated".

This paper examines the optimal pricing and product-pitching strategies for a firm selling goods of varying quality. The focus is on situations where the buyer does not have full information about their preference for product quality and relies on the salesperson for guidance. This guidance is conveyed through *personal selling* — a marketing strategy characterized by direct and interactive communication between the buyer and a company representative. It involves a back-and-forth exchange of information that allows the salesperson to tailor their approach to the buyer's specific needs and inquiries. To analyze such market interactions, I modify the standard monopolistic screening model to incorporate the seller's strategic sequential information disclosure through pre-offer communication with the buyer. More specifically, the seller's interaction with the buyer is divided into two stages: a communication stage and a selling stage. The communication stage proceeds in rounds, where at each round, the buyer privately learns about their ordinal ranking between any two sample products the salesperson chooses to present. With some caveats I outline below, in my setting, this learning technology is equivalent to a sequence of experiments that inform the buyer whether their type is above or below a certain threshold. The seller controls the selection of these thresholds. After each round of communication, the buyer provides feedback to the seller, determining the subsequent course of action. After communication ends, the interaction proceeds to the selling stage, where the seller makes a take-it-or-leave-it offer. The final offer is similarly informative, as the buyer observes if he prefers the offered product to an outside option. Upon seeing the offer, the buyer can accept or reject it but cannot request an alternative. The seller's goal is to determine the most effective pricing and communication strategy, taking into account that the buyer can strategically misreport their learning from the sample offers. I assume the seller has perfect commitment and interpret the resulting seller's problem as the design of an optimal sales manual.

I find that it is optimal for the seller to release information about the thresholds in a bottom-up manner, progressively making the trade-off between product characteristics (quality and price in the baseline version of the model) more extreme. At the optimum, the buyer decides at each instant of communication whether to continue learning about a higher threshold or get to a selling stage immediately. The timing of this transition impacts the quality and price of the product offered, with later transitions corresponding to higher quality and price. This disclosure method strategically obscures information for higher buyer types, who are more likely to misreport, enabling the seller to extract a larger surplus with smaller information rents.

For a linear value-cost version of the model, I show that the insight from Riley & Zeckhauser (1983) no longer holds: a posted price mechanism is suboptimal. Instead, personal selling emerges as a profitable marketing strategy, where the seller communicates and showcases her product before making any offers. By disclosing information in a bottom-up manner, the seller can serve more of the lower types without disrupting the incentives of the higher ones. Intuitively, whenever the buyer observes a positive signal realization about a threshold, they are led to believe they have a higher value for quality. As a result, they are more willing to forgo cheaper product alternatives that are offered at the earlier stages of communication. As the communication process unfolds, many buyer types realize they would have rather stopped communicating sooner, but the respective product offers become unavailable by this point. The seller extracts the whole surplus from the lower buyer types and serves efficient quality to the higher types. To determine the optimal price of this top-quality product, I derive an analog of a virtual type for the learning buyer environment. The virtual surplus in this context demonstrates a blend of forward-looking elements (as seen in Pavan, Segal & Toikka (2014)) due to the sequential nature of the game and backward-looking elements, as the price of the premium quality product influences the incentives for truthful communication among lower types.

Regarding welfare implications, the buyer's expected surplus through an optimal mechanism can either increase or decrease compared to an optimal posted price. I offer a simple sufficient condition for identifying when the absence of information and the presence of communication might disadvantage the buyers. Remarkably, communication is hurtful to the buyers whenever the consumer surplus under an optimal posted price exceeds the corresponding deadweight loss.

I would like to emphasize that the learning mechanism in this paper imposes the most extreme version of a salesperson's control over a buyer's experience and learning: the buyer learns only from *concrete* information that is explicitly presented in the store (Slovic (1972), Bettman & Kakkar (1977)). In particular, I assume the buyer can only rank the products presented to them as part of the same menu by the seller, and the resulting ranking is a unique source of information to them. Consequently, this means that the buyer cannot exactly estimate their precise willingness to pay from a single interaction with a product or compare the currently observed products with the ones they have seen before. Admittedly, such severe limitations on buyers' learning can be realistic only for very particular applications. For other applications, it is more prudent to view my model as a benchmark, illustrating what the seller could accomplish by strategically selecting which information becomes accessible to the buyer over time. I present preliminary observations regarding the potential impact of different assumptions about the buyer's experience on the model's predictions. However, a comprehensive analysis and characterization of the optimal mechanism under these alternative assumptions is beyond the scope of the current project.

## 1.1 Related Literature

My paper combines elements of sequential screening/mechanism design (Courty & Hao (2000), Nocke, Peitz & Rosar (2011), Pavan, Segal & Toikka (2014)) with sequential information design (e.g. Doval & Ely (2020), Che & Hörner (2017)). Bergemann & Wambach (2015) consider a similar problem in an auction setting of a single good allocation. Analogously, they find that bottom-up information disclosure is optimal for an auctioneer. Since the auctioneer allocates a single good, all exiting buyer types receive a zero outcome. Consequently, if the consumer has no private information at the beginning of the interaction, the seller can extract the full market surplus: no distortion is required for screening purposes. Thus, both papers highlight the advantage of keeping higher types uninformed for as long as possible due to their stricter incentive constraints. In Ező & Szentes (2007) and Wei & Green (2022), the seller dynamically screens buyers with different prior information and controls what the buyers can learn about their preferences. Importantly, in my setting, the seller can also discriminate different buyer types with the quality of an offered product. Another relevant study by Ostrizek & Shishkin (2022) analyzes dynamic interaction between a buyer and a seller but from a different perspective. They focus on the temporal evolution of preferences rather than the acquisition of new information. In particular, the monopolist strategically frames each choice made by the consumer to exploit their time inconsistency.

The most closely related paper is the one by Bergemann, Heumann & Morris (2022), which considers a combined problem of Bayesian persuasion from Kamenica & Gentzkow (2011) and monopolistic screening from Mussa & Rosen (1978). They consider a much broader information strategy space but find that the optimal solution involves the seller dividing the buyer types into finitely many intervals, with the buyer learning the specific interval corresponding to their true type. My paper deviates from Bergemann, Heumann & Morris (2022) in two ways. Firstly, I incorporate sequential information provision, enabling the seller to better manage the buyer's incentives. Secondly, my model imposes ex-post participation constraints instead of interim ones, as the final offer provides new information to the buyer. This assumption implies that a simple posted price mechanism can no longer achieve a full surplus extraction in a standard linear setting.

Additionally, with convex seller's costs as in Mussa & Rosen (1978) set-up, I establish that the optimal information strategy avoids pooling any served buyer types, in contrast to the findings of Bergemann, Heumann & Morris (2022).

The remainder of the paper is organized as follows. Section 2 introduces the basic setup of the model. In Section 3, I present an illustrative example to elucidate the key elements of the model and anticipated results. Section 4 provides a comprehensive formal depiction of the model, explaining the space of the seller's feasible strategies. In Section 5, I posit the paper's main result in Theorem 1 and highlight its essential characteristics. Section 5.3 provides an overview of the proof of Theorem 1. I discuss the main assumptions of the model in Section 6, and consider model extensions in Section 6. Section 9 concludes.

# 2 Basic Set-Up

A seller S (she) produces a good of varying quality $q \in [0, 1]$ and offers it to a buyer B (he) of an unknown preference type $\theta$. The buyer type is drawn from a finite set $\Theta \subset \mathbb{R}_+$ according to a full support prior distribution $\mu_0 \in \Delta(\Theta)$. A buyer of type $\theta$ purchasing quality $q$ at price $p$ derives utility $v(\theta, q, p) = \theta q - p$. The seller produces quality at a constant marginal cost $c \in \mathbb{R}_+$. I discuss more general cost and preference specifications in Section 4. The buyer's type $\theta$ is initially unobservable to both parties. Instead, the buyer learns about his type through the interaction with the seller, which I divide into two stages: a communication stage and a selling stage.

The communication stage is divided into periods. In each period, the buyer privately learns if his type is above a threshold $\tau \in \mathbb{R}$, which the seller chooses. Formally, given a threshold $\tau$, the buyer's true type $\theta$ is mapped to a deterministic signal realization $\sigma : \Theta \times \mathbb{R} \to \{a, b\}$:

$$\sigma(\theta|\tau) = \begin{cases} a(bove), & \text{if } \theta > \tau \\ b(elow), & \text{if } \theta \leq \tau \end{cases}$$

After observing the signal realization, the buyer sends one of the two messages $M = \{m^a, m^b\}$.

The seller can make the communication stage as long as she wants, and information provision is completely free (in particular, there is no discounting between the periods). After communication ends, the interaction proceeds to a selling stage, where the seller presents a single take-it-or-leave-it offer to the buyer. I assume that this offer provides another signal to the buyer. Specifically, after the take-it-or-leave-it offer of quality $q$ at a price $p$ is presented, the buyer observes a signal realization according to $\sigma(\theta|q/p)$.

I interpret information technology as follows. During the communication stage, the seller offers a binary menu of sample products $\{(q^1, p^1), (q^2, p^2)\}$ to the buyer. Upon seeing the menu, the buyer learns only the ordinal ranking between the products (determined by his true type $\theta$). Specifically, he learns whether $v(\theta, q_1, p_1) > v(\theta, q_2, p_2)$ or not, which is equivalent to whether $\theta > (p_2 - p_1)/(q_1 - q_2)$.[1] Similarly, in the selling stage, the buyer learns from a binary menu $\{(q, p), (0, 0)\}$ after getting a final offer of $(q, p)$. Importantly, at every stage of interaction, the buyer cannot learn from *any* menus that are not directly shown to him: *concrete* evidence is required. As suggested by Slovic (1972), a decision-maker uses only the information that is explicitly presented to them. Further details and discussion of the concrete evidence assumptions are provided in Section 6.

The seller designs and perfectly commits to an extensive form that describes how communication should proceed for every possible buyer's response. This can be understood as the owner of a shop creating a detailed manual for their salespeople, specifying how they should interact with the buyer in different scenarios. It is assumed that the buyer comprehends the extensive form and can anticipate how their actions impact future information and product offers. To enhance clarity and facilitate understanding, I next provide some illustrative examples that highlight the key assumptions and anticipated outcomes of the model. I postpone to Section 4 the formal details regarding the decision problems faced by each agent.

# 3 Examples and Anticipated Results

To provide a concrete illustration, let's consider a situation where the seller offers enterprise software with varying speeds $q \in [0, 1]$, at a zero marginal cost ($c = 0$). The potential buyers have

[1] The analysis would remain the same if we instead assumed that there is another signal realization for buyer's indifference between the products.

varying software speed requirements, represented by $\theta$. This "taste" parameter could represent factors like the number of transactions the software is expected to handle within a client firm. Before formalizing a long-term contract, the seller allows the buyer to acquire different demonstration versions of the software for a brief trial period.[2] Crucially, the client firm can only use one demo version at a time and can only determine whether the current version justifies its price. In particular, they cannot directly compare different samples amongst themselves. Following the buyer's feedback on their usage experience, the seller presents a take-it-or-leave-it offer.

Even though I consider a finite-type model, I assume $\theta \sim U[0,1]$ to make illustrations cleaner. The main takeaways do not rely on the continuous distribution of the buyer's type.

**Example 1.1** (Posted Price with No Communication)**.** First, suppose the seller gives up on the communication stage and makes a single offer $(q, p)$. The game unfolds as follows. Upon seeing the offer, the buyer observes a signal realization $\sigma(\cdot|q/p)$. Given the signal realization, he then decides whether to accept the offer. It is evident that buyer types who observe a signal realization "above" strictly prefer to purchase, whereas buyers with a realization "below" optimally choose to reject the offer. Thus, without communication, the posted price mechanism yields the same outcome as the standard model of a privately informed buyer. In particular, the seller can implement an optimal posted price mechanism with the offer $(q = 1, p = 1/2)$ yielding an expected profit of 0.25.

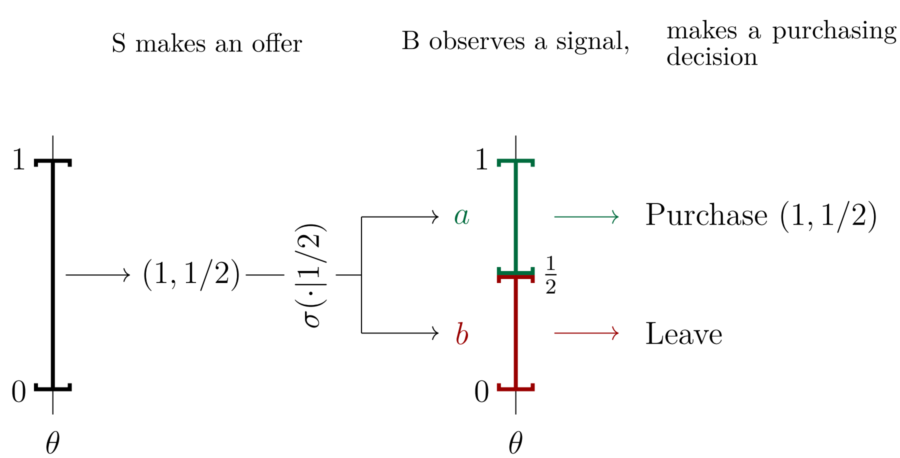



*Figure 1: Optimal Posted Price Mechanism*

[2]It is assumed that, relative to the final contract terms, the trial period is sufficiently short so as the trial period profit is negligible.

**Example 1.2** (One Pre-Offer Sample)**.** Even very short communication allows the seller to increase her profit. Assume the seller offers one sample product to the buyer before making a final offer, and the sample product reveals to the buyer whether his type is above or below $\frac{1}{2}$. In period two, communication ceases. If the buyer claims to have observed a signal "above" the seller offers $\left(1, \frac{1}{2}\right)$, otherwise she proposes $\left(\frac{1}{2}, \frac{1}{8}\right)$. Figure 2 below depicts the game's outcome.

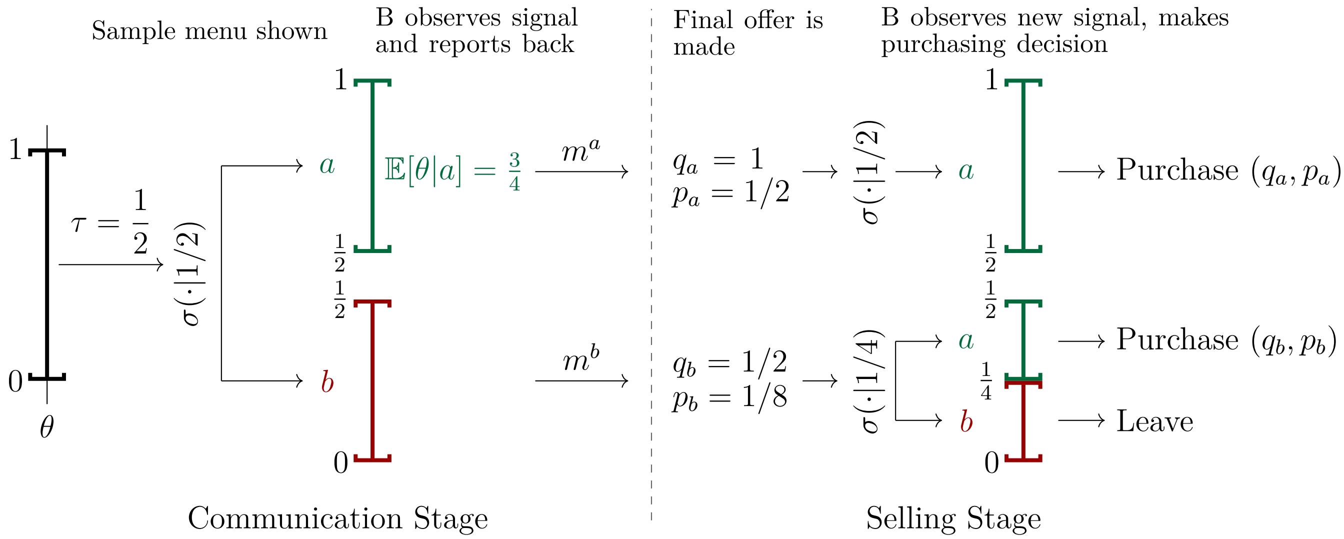



*Figure 2: One Pre-Offer Sample Mechanism*

*Note*: The figure depicts the outcome of a mechanism where the seller offers one sample before making a final offer. The sample informs the buyer about a threshold 1/2: buyer types in $(1/2, 1]$ observe a signal realization $a$, and $[0, 1/2]$ — observe $b$. In the depicted outcome, the buyer is truthful, so the buyer types $(1/2, 1]$ report $m^a$ and receive an offer $(1, 1/2)$, while the buyer types $[0, 1/2]$ report $m^b$ and are offered $(1/2, 1/8)$. In each case, the final offer further informs the buyer if it is preferable to an outside option. Furthermore, all buyer types within $(1/2, 1]$ realize that they prefer to purchase $(1, 1/2)$ and consequently accept the offer. Similarly, buyer types $(1/4, 1/2]$ recognize their preference for purchasing $(1/2, 1/8)$ and therefore accept the corresponding offer. In contrast, buyer types within $[0, 1/4]$ learn that they would rather decline the purchase, leading them to reject the offer.

Notice that this mechanism improves upon the optimal posted price of 1/2: buyer types in $[1/2, 1]$ purchase the same good, while there is an additional share of the market $[1/4, 1/2]$ who now purchase a lower-quality product. The total profit increases to $\approx 0.281$. Let us verify that the buyer is willing to communicate truthfully and has proper incentives to report $m^s$ after observing signal $s$ in the communication stage. Consider a report choice by a buyer who observes realization "above" after the sample product is shown. He foresees that a truthful report leads to a purchase of $(q_a, p_a)$ and a misreport — to a purchase of $(q_b, p_b)$. He then compares the consequences of each report, taking into account the information from the sample. In the example above, the numbers are chosen so that the buyer observing "above" is exactly indifferent between either report:

$$\underbrace{\mathbb{E}\left[v\left(\theta, q_a, p_a\right)\middle| a\right]}_{\text{Continuation Value of Truthful Report}} = \mathbb{E}[\theta|a] \cdot q_a - p_a = \frac{3}{4} \cdot 1 - \frac{1}{2} = \frac{1}{4}$$

$$\underbrace{\mathbb{E}\left[v\left(\theta, q_b, p_b\right)\middle| a\right]}_{\text{Continuation Value of Misreporting}} = \mathbb{E}[\theta|a] \cdot q_b - p_b = \frac{3}{4} \cdot \frac{1}{2} - \frac{1}{8} = \frac{1}{4}$$

It's worth noting that the assumption of a buyer requiring concrete evidence is critical for the mechanism to hold together. The buyer correctly anticipates which offers are made after each possible report but cannot extract any information from these products without seeing them. If the buyer could analyze his preferences for the two possible final offers in period 1, without actually seeing the menu $\{(q_a, p_a), (q_b, p_b)\}$, he could additionally learn whether his type is above or below $(q_a - q_a)/(p_a - p_b) = 3/4$. This would enable him to make a more informed decision about whether to lie in the communication stage and all the buyer types in $[1/2, 3/4]$ would then prefer to lie.

**Example 1.3** (Bottom-Up Communication)**.** In Example 1.2, the seller improves upon profit by serving an additional segment of customers. The offer for this new segment $(q_b, p_b)$ is constructed to preserve higher types' incentives to purchase a more expensive version of the product. I will now show how the seller can further increase her profit by continuously screening the whole segment of the market that is not served under the optimal posted price mechanism.

Suppose that communication continues over time in $[0, 1/2]$. Consider the following mechanism: at each period $t \in [0, 1/2]$, the seller reveals to the buyer if he is above or below $\tau_t = t$. The buyer decides at every instance whether to continue to get more information or terminate and get an offer. If communication ends at period $t$, the buyer gets an offer $(q_t, p_t)$. I next specify the offers to make sure the buyer is willing to end communication at the first instance of a "below" signal. I depict this game's outcome in Figure 3.

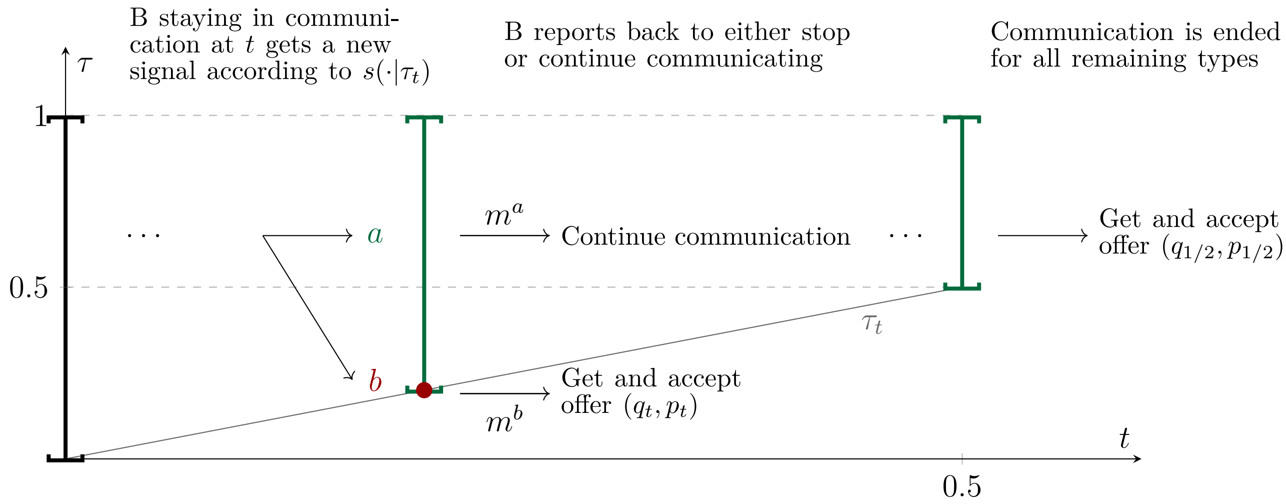



*Figure 3: Bottom-Up Communication*

*Note*: The figure summarizes bottom-up communication. In period $t$, the buyer learns if his type is above $\tau_t = t$. The figure depicts an outcome where the buyer terminates communication after getting the first negative signal. In period $t$, the lowest remaining type learns his type and leaves communication to get an offer $(q_t, p_t)$. The buyer types $(t, 1]$ learn they are above $t$ and stay in communication further. In period 1/2, only types $[1/2, 1]$ are remaining, and they all get a premium offer $(1, 1/2)$.

Note that given the choice of threshold for every instance of communication, at period $t$ the buyer believes his type is in $\hat{\Theta}_t$ with

$$\hat{\Theta}_t = \begin{cases} t, \text{ if } s_t = b \text{ and } s_{t'} = a, \forall t' < t \\ (t, 1], \text{ if } s_{t'} = a, \forall t' \leq t \end{cases}$$

Construct the offers for each period $t$ as follows:

$$q_t = \frac{1/4}{(1-t)^2} \qquad p_t = \frac{t/4}{(1-t)^2} \tag{1}$$

These offers satisfy the following properties: (i) $p_t$ extracts all the surplus from a type that currently observes the first signal realization "below", or $p_t = t \cdot q_t$; (ii) the buyer who does not observe a signal "below" by (and including) period $t$ is indifferent between continuing communication and purchasing $(q_t, p_t)$ right away; (iii) all the types in $(1/2, 1]$ purchase the same product as in the optimal posted price mechanism.

Let us verify that condition (ii). A buyer who has only seen signal realizations "above" by period $t$ is just willing to stay in communication whenever his expected continuation value is the same as the expected value from purchasing $(q_t, p_t)$. Since all the buyer types below 1/2 get their full surplus extracted, the only value of communication comes from the buyer's expectation to be some type in $(1/2, 1]$ who all purchase $\big(q_{1/2} = 1, p_{1/2} = 1/2\big)$.

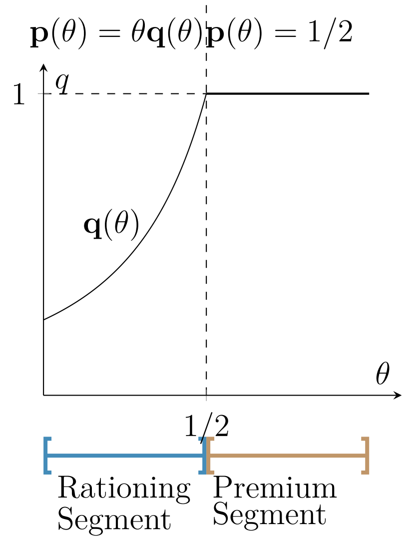
$$\underbrace{\Pr\left(\theta \in (1/2,1] \big| \theta \in \hat{\Theta}_t\right) \cdot \mathbb{E}\big[v(\theta, q_{1/2}, p_{1/2}) | \theta \in (1/2,1]\big]}_{\text{Continuation Value of Communication}} = \frac{1/2}{1-t} \cdot \left(\frac{3}{4} \cdot 1 - \frac{1}{2}\right) = \frac{1/8}{1-t} \tag{2}$$

$$\underbrace{\mathbb{E}[v(\theta, q_t, p_t) | \theta \in \hat{\Theta}_t]}_{\text{Expected Value of Purchasing } (q_t, p_t)} = \mathbb{E}[\theta | \theta > t] \cdot q_t - p_t = \frac{1+t}{2} \cdot q_t - t q_t = \frac{1-t}{2} \cdot q_t \tag{3}$$

The two get equated exactly when $q_t$ is as specified in Equation (1). It is easy to see that whenever the buyer observes the signal "below", he prefers to leave immediately, as he foresees the future offers to be too expensive given his discovered type. Thus, it is incentive compatible for a buyer to immediately end communication purchasing $(q_t, p_t)$ after the first "below" signal realization and to proceed with communication if no such signal is observed.

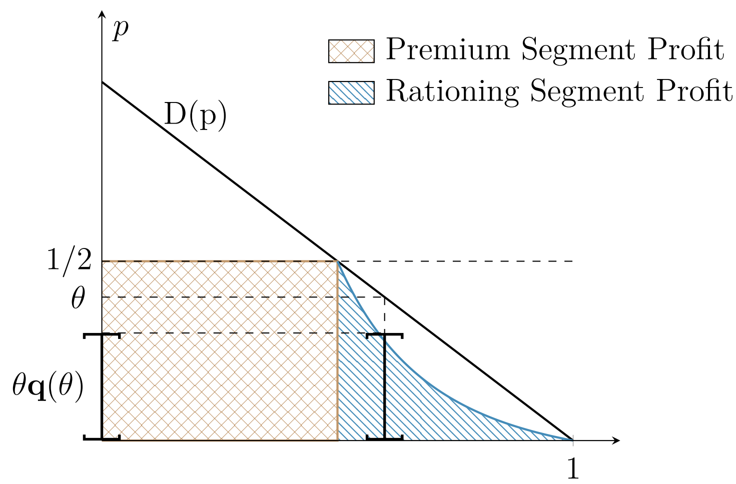
In the induced allocation, all buyer types in $[0, 1/2]$ are perfectly screened with strictly increasing rationed quality offers and get all their surplus extracted. I refer to such types as a *rationing segment* of the market. All the buyer types in $(1/2, 1]$ belong to a *premium segment* and purchase the highest feasible quality at a constant price. The seller's total profit in this mechanism is:

$$\underbrace{\int_0^{1/2} \mathbf{p}(\theta) d\theta}_{\text{Rationing segment profit}} + \underbrace{\frac{1}{2} \cdot p_{1/2}}_{\text{Premium segment profit}} = \frac{1}{4}\left(\log(1/2) + 1\right) + \frac{1}{4} \approx 0.326$$




*(a) Induced Allocation*

*Note*: The figure depicts the induced allocation that each type $\theta$ purchases in the bottom-up communication of length 1/2.




*(b) Seller's Total Profit*

*Note*: The figure summarizes the seller's profit in a bottom-up communication of length 1/2.

**Example 1.4** (Optimal Segmentation in Bottom-Up Communication)**.** Finally, the seller can further improve her profits by optimally segmenting the market. From the previous example, the consumer surplus of the premium segment forms the continuation value of proceeding with communication. Hence, the higher this surplus, the smaller rationing is required to prevent the buyer from abandoning communication too soon. Keeping the main structure the same as in the previous example, suppose the seller screens types in the rationing segment by communicating over $[0, \tilde{\tau}]$ for some $\tilde{\tau} \in [0, 1]$. If the buyer stops communication in period $t$, he gets an offer $(q_t, p_t)$ with:

$$q_t = \frac{(1 - \tilde{\tau})^2}{(1 - t)^2} \qquad p_t = t\frac{(1 - \tilde{\tau})^2}{(1 - t)^2}$$

The offers are again designed to satisfy conditions (i)-(ii) as in Example 1.3. The buyer of type $\theta$ then leaves communication at period $\theta$ and ends up paying $\theta \cdot (1 - \tilde{\tau})^2/(1 - \theta)^2$. The types belonging to a premium segment $(\tilde{\tau}, 1]$ stay in communication until $\tilde{\tau}$ and purchase quality 1 at a price of $\tilde{\tau}$.

Seller's expected profit in a bottom-up communication with segmentation $\tilde{\tau}$ can be expressed as

$$\Pi(\tilde{\tau}) = \underbrace{\int_0^{\tilde{\tau}} \theta \frac{(1-\tilde{\tau})^2}{(1-\theta)^2} d\theta}_{\text{Profit from the rationing segment}} + \underbrace{(1-\tilde{\tau}) \cdot \tilde{\tau}}_{\text{Profit from the premium segment}}$$

The seller balances two key effects when choosing the segmentation in a bottom-up screening mechanism. On the one hand, she can charge a higher price in a smaller premium segment. But on the other hand, a smaller premium segment implies that the incentives for staying in communication get weaker. To compensate for this effect, the seller must decrease the price (and quality) for all consumers in the rationing segment.

$$\text{Total Marginal Effect of } (\uparrow \tilde{\tau}) = \underbrace{(1-\tilde{\tau})}_{\text{Price effect on the premium market}} \underbrace{- \int_0^{\tilde{\tau}} \frac{\theta(1-\tilde{\tau})}{(1-\theta)^2/2} d\theta}_{\text{Quality effect in the rationing segment}}$$

Observe that the two effects are perfectly balanced at a point where $\int_0^{\tilde{\tau}} \theta / \left[(1-\theta)^2/2\right] d\theta$ crosses 1. In Section 5, I explain that this expression can be considered an analog of a virtual type for a learning buyer environment.

Maximizing over $\tilde{\tau}$, one gets that the optimal segmentation is given by $\tilde{\tau}^* \approx 0.576$. The expected profit $\approx 0.334$. Theorem 1 establishes that for any model parameters, the seller cannot achieve any higher profit than the one generated by bottom-up communication with the optimal segmentation. It further states that the optimal segmentation is given by an intersection of a learning buyer's virtual type with 1.

# 4 Full Model

In this section, I present a comprehensive exposition of the model, detailing the decision-making processes of both the seller and the buyer. The section concludes with a formulation of the seller's problem. A reader primarily interested in the paper's main results may choose to proceed directly to Section 5. However, the notation introduced in this section will be referenced and utilized when proving Theorem 1 in Section 5.3.

The seller designs an extensive form, denoted by a set of seller-observed histories $H^S$, which captures how communication proceeds and which final offers are made in different communication scenarios. The message space $M$ in the communication stage is fixed to be a pair $\{m^a, m^b\}$[3]. The seller-observed histories are divided into communication stage histories, denoted as $H^S_C$, and selling stage histories, denoted as $H^S_O$. A communication stage history for the seller after $k$ rounds $h^S_k = (\boldsymbol{\tau}_k, \mathbf{m}_{k-1})$ consists of a sequence of $k$ thresholds $\boldsymbol{\tau}_k \in \mathbb{R}^k$ and a sequence of $k-1$ reports $\mathbf{m}_{k-1} \in M^{k-1}$. A selling stage history which follows $k$ rounds of communication $h^S_{k,o} = (\boldsymbol{\tau}_k, \mathbf{m}_k, q, p)$ contains sequences of $k$ thresholds and buyer reports, and a take-it-or-leave-it offer of quality and price $(q,p) \in [0,1] \times \mathbb{R}$. It is assumed that $\emptyset \in H^S$, and if $H^S$ includes some history $h^S$, it also includes all its subhistories.

For every seller-observed history in the communication stage $h^S_k \in H^S_C$, and every report at stage $k$, the extensive form $H^S$ must describe how the game proceeds. The seller has two options: continue communication by introducing a new threshold $\tau_{k+1}$, or conclude communication and move to the selling stage with an offer $(q,p)$. Formally, for every $h^S_k = (\boldsymbol{\tau}_k, \mathbf{m}_{k-1}) \in H^S_C$ and every $m \in M$, exactly one is true: either $h^S_{k+1} = ((\boldsymbol{\tau}_k, \tau_{k+1}), (\mathbf{m}_{k-1}, m)) \in H^S_C$ for some unique $\tau_{k+1} \in \mathbb{R}$, or $h^S_{k,o} = (\boldsymbol{\tau}_k, (\mathbf{m}_{k-1}, m), q, p) \in H^S_O$ for some unique $(q,p) \in [0,1] \times \mathbb{R}$. Once the seller makes an offer, the interaction with the buyer ends, meaning that all selling stage histories are terminal in $H^S$. Let $\mathcal{H}^K$ be the set of all possible extensive forms where communication does exceed $K$ rounds, and $\mathcal{H} = \cup_{K=1}^{\infty} \mathcal{H}^K$ — the set of all possible extensive forms.

Given the seller's choice of the extensive form $H^S \in \mathcal{H}$, define $H^B = H^B_C \cup H^B_O$ be the set of plausible buyer observed histories. In these histories, the buyer has access to past thresholds, reports, and signal realizations but lacks direct knowledge of his own type $\theta$. During the communication stage, after $k$ rounds, the buyer observes a history denoted as $h^B_k = (\boldsymbol{\tau}_k, \mathbf{m}_{k-1}, \mathbf{s}_k)$, which contains all past thresholds and reports, as well as the sequence of signal realizations $\mathbf{s}_k \in (\{a,b\})^k$. In the selling stage, after $k$ instances of communication, the buyer observes $k$ thresholds, reports, a final product offer by the seller, and $k+1$ signal realizations. I denote a buyer-observable selling

[3]By the revelation principle, this is without loss of optimality.

stage history after $k$ rounds as $h^B_{k,o} = (\boldsymbol{\tau}_k, \mathbf{m}_k, q, p, (\mathbf{s}_k, s_o))$. with $s_o \in \{a, b\}$ representing the signal observed from the final offer.

Buyer-observed history is *plausible* if a respective seller-observed history $h^S_k = (\boldsymbol{\tau}_k, \mathbf{m}_k)$ or $h^S_{k,o} = (\boldsymbol{\tau}_k, \mathbf{m}_k, q, p)$ is part of the extensive form $H^S$ and if the observed history is consistent with some buyer type $\theta$. The consistency with type $\theta$ means that the observed signal realizations $\mathbf{s}_k$ during communication agree with $\sigma(\theta|\boldsymbol{\tau}_k)$, and the final signal realization $s_o$ agrees with $\sigma(\theta|q/p)$ for selling stage histories. I also use the notation $H^{B,\theta}_C$, $H^{B,\theta}_O$ to denote plausible buyer histories in a respective stage that are consistent with $\theta$. For every plausible buyer-observable history, let $\hat{\Theta}(h^B) \equiv \{\theta : h^B \in H^{B,\theta}\}$ denote the set of types consistent with such history.

A buyer's pure strategy $\beta : H^B \to M \cup \{0, 1\}$ [4] specifies the buyer's action for every plausible buyer-observed history. In the communication stage, the strategy determines which message from $M$ is sent to the seller ($\beta(H^B_C) \subseteq M$), while in the selling stage, it determines the purchasing decision ($\beta(H^B_O) \subseteq \{0, 1\}$) [5]. Let $\mathcal{B}_{H^S}$ denote the set of all feasible pure buyer strategies in an extensive form $H^S$.

Define $\omega^{H^S_;\beta} : \Theta \to H^B$ to be an *outcome* of buyer's strategy $\beta$ for every possible buyer's type. Formally, $\omega^{H^S_;\beta}(\theta)$ is a selling stage history $h^B_{k,o} = (\boldsymbol{\tau}_k, \mathbf{m}_k, q, p, \mathbf{s}_k, s_o) \in H^{B,\theta}_O$ consistent with type $\theta$, such that the strategy $\beta$ at every subhistory for every subhistory $(\boldsymbol{\tau}_l, \mathbf{m}_{l-1}, \mathbf{s}_l)$ agrees with $h^B_{k,o}$: $\beta((\boldsymbol{\tau}_l, \mathbf{m}_{l-1}, \mathbf{s}_l)) = m_l$. In addition, for the respective elements of an outcome, let $\mathbf{q}^{H^S_;\beta} : \Theta \to [0, 1], \mathbf{p}^{H^S_;\beta} : \Theta \to \mathbb{R}$ denote *outcome allocations* of quality and price that get offered to type $\theta$ on the path of the game.

To summarize, the seller chooses an extensive form game $H^S = H^S_C \cup H^S_O$ consisting of communication and selling stage histories. During communication, the seller provides an experiment to the buyer, where he learns if his type is above some threshold. The buyer's learning in this experiment is private. After every seller-observable communication stage history, the extensive form describes how the interaction with the buyer proceeds with each of the buyer's potential reports in $M = \{m^a, m^b\}$. For every report, either communication moves forward, and the buyer further learns whether his type is above some other threshold, or the buyer gets to a selling stage,

[4] Again, buyer's strategy is deterministic without loss of optimality by the revelation principle.

[5] Where 0 stands for "no purchase" action.

where he gets some quality-price offer. Given the extensive form $H^S$, the buyer chooses a pure strategy $\beta$, which specifies a report during the communication stage and a purchasing decision at the selling stage. The buyer's strategy depends on the information available to the buyer at each point of time, which consists of past thresholds, reports, signal realizations (and a final offer if at the selling stage). Given a pair of an extensive form $H^S$ and buyer's pure strategy $\beta$, every type $\theta$ gets to a unique selling stage history — outcome — which I denote $\omega^{H^S_,\beta}(\theta)$ and gets an outcome allocation $\mathbf{q}^{H^S_,\beta}(\theta), \mathbf{p}^{H^S_,\beta}(\theta)$.

The seller's payoff $V^S$ and the buyer's payoff $V^B$ given a pair of an extensive form $H^S$ and buyer's strategy $\beta$ are, respectively, profit and utility from an outcome allocation if the purchase is made at the outcome (and zero otherwise):

$$V^B(H^S_,\beta) = \sum_{\theta\in\Theta} v(\theta, \mathbf{q}^{H^S_,\beta}(\theta), \mathbf{p}^{H^S_,\beta}(\theta)) \cdot \beta(\omega^{H^S_,\beta}(\theta)) \cdot \mu_0(\theta)$$

$$V^S(H^S_,\beta) = \sum_{\theta\in\Theta} (\mathbf{p}^{H^S_,\beta}(\theta) - c \times \mathbf{q}^{H^S_,\beta}(\theta)) \cdot \beta(\omega^{H^S_,\beta}(\theta)) \cdot \mu_0(\theta)$$

Finally, the Seller's Problem is a choice of a *mechanism*, consisting of an extensive form $H^S$ and a buyer-optimal strategy $\beta$, that maximizes the seller's payoff:

**Seller's Problem:**

$$\max_{H^S\in\mathcal{H}, \beta\in\mathcal{B}_{H^S}} V^S(H^S_,\beta)$$

$$\text{subject to } V^B(H^S_,\beta) \geq V^B(H^S_,\beta'), \forall \beta' \in \mathcal{B}_{H^S}$$

In the next section, I characterize an optimal solution to the above problem.

# 5 Seller's Optimal Mechanism

In this section, I present the main result of the paper, which describes an optimal solution to the Seller's Problem in Theorem 1. I establish in Lemma 2 the optimum has the features of Example 1.4: the seller uses a bottom-up communication, meaning the seller sequentially reveals information to the buyer by incrementally increasing a threshold. At every instance of bottom-up communication, the buyer makes a decision whether he wants to continue or terminate communication by proceeding to a selling stage. The later the buyer exits communication, the higher the quality and corresponding price of the offer. If the buyer remains in communication until the last round, he is presented with a premium quality offer. The mechanism's outcome divides buyer types into two segments based on whether they purchase a premium quality offer or a rationed quality one. Within the rationing segment, the seller extracts the whole buyer's surplus by offering a different rationed-quality product to each included type. Conversely, the premium segment groups higher types by offering them the same premium deal. Furthermore, I introduce the concept of a learning-buyer virtual type, which determines the optimal market segmentation.

In Section 5.3, I provide an overview for the proof Theorem 1 by focusing on specific deviations by the buyer. In particular, I only allow the buyer to deviate from a given strategy by mimicking some type he believes to be lower than a true one. I show that the mechanism of Theorem 1 is optimal against such deviations.

## 5.1 Bottom-Up Communication and Rationing

For convenience, let $\Theta = \{\theta_1, \dots, \theta_n\}$ with $\theta_{i+1} > \theta_i$. I assume that $\theta_n > c$, ensuring that it is strictly profitable to serve at least the highest type.[6] Now, I describe the main features of an optimal mechanism.

I begin by describing optimal communication first. *Bottom-up communication* proceeds as follows: at stage $i$, the seller reveals whether the buyer's type is above $\theta_i$. The buyer decides whether to continue with the communication process. If the buyer decides to leave at stage $i$, he is offered $(q_i, p_i)$. If the buyer does not leave after the last round, he receives a premium offer

[6] Otherwise, the seller can achieve first-best by offering zero product to a buyer with no communication.

$(q^{pr}, p^{pr})$.

**Definition 1** (Bottom-Up Communication)**.** An extensive form $H^S$ features bottom-up communication of length $K$ if:

(i) Communication never exceeds $K \leq n-1$ rounds.

(ii) At round $i$ of the communication stage, the buyer learns if his type is above the threshold $\theta_i$.

(iii) After each communication stage history, the buyer can either stay in the communication process (and learn about $\theta_{i+1}$ in the next period) or move to a selling stage where he gets an offer $(q_i, p_i)$.

(iv) If the buyer does not leave after $K$ rounds of communication, he gets a premium product offer $(q^{pr}, p^{pr})$.

Construction of the final offers for the selling stages mirrors the method illustrated in Example 1.3. As I verify later, it ensures the buyer is willing to cease communication after the first negative signal. Consequently, every buyer type $\theta_i$ with $i < K$ abandons communication at exactly round $i$. The offer $(q_i, p_i)$ presented after this round contains some rationed quality $q_i < 1$ and extracts the entire surplus of the exiting buyer type $\theta_i$. Additionally, if the quality $q_i$ is positive, the buyer with a positive signal about the threshold $\theta_i$ is indifferent between either action in round $i$. Specifically, the rationing of $q_i$ is adjusted to equalize the two surpluses: 1) the anticipated surplus from continuing communication and 2) the expected surplus from moving to the selling stage to secure an offer $(q_i, p_i)$. This rationing scheme is referred to as *surplus-based rationing.*

**Definition 2** (Surplus-Based Rationing)**.** Given an extensive-form $H^S$ with bottom-up communication of length $K$, say that the final offers $\left\{(q_i, p_i)_{i=1}^K, (q^{pr}, p^{pr})\right\}$ feature surplus-based rationing if the premium offer is $q^{pr} = 1, p^{pr} = \theta_{K+1}$ and for all $i \leq K$:

$$q_i = \frac{\mathbb{E}_\theta[(\theta - \theta_{K+1})_+]}{\mathbb{E}_\theta[(\theta - \theta_i)_+]}, \; p_i = \theta_i \cdot q_i, \; \text{whenever } \theta_i > c$$

$$q_i = 0, \; p_i = 0, \; \text{whenever } \theta_i \leq c$$

I now verify that every type $\theta_i \leq \theta_K$ is willing to leave at round $i$ and purchase offer $(q_i, p_i)$, while every type $\theta_i > \theta_K$ is willing to stay in communication after round $K$ and purchase $(q^{pr}, p^{pr})$. Note that if a buyer receives his first negative signal in round $i$, he discovers his type is exactly $\theta_i$. If such a buyer leaves at round $i$, he gets zero utility as $(q_i, p_i)$ extracts his entire surplus. Alternatively, he can use a sequence of reports that either leads to some offer $(q_j, p_j)$ for $j > i$, or to a premium offer. However, any such deviation is not profitable, as either of these offers extracts the whole surplus from an even higher type.[7] Suppose now the buyer gets a positive signal about threshold $\theta_i$. In this case, the continuation value of communication is the buyer's conditional expected consumer surplus from purchasing a premium offer. Thus, the continuation value of communication is

$$\frac{\Pr(\theta > \theta_{K+1})}{\Pr(\theta > \theta_i)} \mathbb{E}_\theta[(\theta - \theta_{K+1})|\theta > \theta_{K+1}] = \frac{\mathbb{E}_\theta[(\theta - \theta_{K+1})_+]}{\Pr(\theta > \theta_i)}$$

As an alternative, the buyer may cease communication immediately and accept an offer $(q_i, p_i)$, resulting in a payoff

$$\mathbb{E}[(\theta q_i - p_i|\theta > \theta_i] = q_i \cdot \frac{\mathbb{E}_\theta[(\theta - \theta_i)_+]}{\Pr(\theta > \theta_i)}$$

Note that if $q_i$ is surplus-based rationed, the buyer with a positive signal about $\theta_i$ is just willing to continue communication, as intended. This verifies that bottom-up communication is incentive-compatible, with the buyer leaving communication after the first negative signal when the offers satisfy surplus-based rationing. Bottom-up communication (with surplus-based rationed final offers) results in the market getting segmented into two parts: lower buyer types decide to cease communication before the last round and get sorted into the *rationining* segment, where they get their whole surplus extracted. Meanwhile, higher buyer types choose to stay after the last round and get assigned to the *premium segment*. Next, I determine the optimal premium segment.

[7] $v(\theta_i, q_j, p_j) = q_j \cdot \theta_i - p_j = q_j \cdot \theta_i - q_j \cdot \theta_j < 0$ and $v(\theta_i, q^{pr}, p^{pr}) = q^{pr} \cdot \theta_i - p^{pr} = \theta_i - \theta_{K+1} < 0$.

**Definition 3.** Define a *learning-buyer (normalized) virtual type* $\gamma^{lb} : \Theta \to \mathbb{R} \cup \infty$

$$\gamma^l(\theta_i) \equiv \sum_{j=1}^{i} \frac{\max\{\theta_j - c, 0\}}{\mathbb{E}_\theta[(\theta - \theta_j)_+]} \mu_0(\theta_j)$$

As hinted in Example 1.4, the learning buyer virtual type captures the effects of slightly expanding the premium segment. To elaborate, let me consider bottom-up communication of length $i+1$, where surplus-based rationing is applied for the final offers. Now, let me examine the consequences of shifting type $\theta_i$ to the premium segment. By the definition of surplus-based rationing, the offer quality after each earlier round $j < i+1$ (if positive) increases by:

$$\Delta q_j = \frac{(\theta_{i+1} - \theta_i) \Pr(\theta > \theta_i)}{\mathbb{E}_\theta[(\theta - \theta_j)_+]}$$

This leads to a cumulative increase in the rationing segment profit by:

$$\Delta \Pi^{rat} = \sum_{j=1}^{i} \frac{\max\{\theta_j - c, 0\} \mu_0(\theta_j)}{\mathbb{E}\theta[(\theta - \theta_j)_+]} \Delta q_j$$

On the other hand, in the premium segment, the price decreases, resulting in a profit decrease of

$$\Delta \Pi^{pr} = -(\theta_{i+1} - \theta_i) \Pr(\theta > \theta_i)$$

. The total change in profit from the two segments is then:

$$\Delta \Pi = \Delta \Pi^{rat} + \Delta \Pi^{pr} = [(\theta_{i+1} - \theta_i) \Pr(\theta > \theta_i)] \left[\gamma^l(\theta_i) - 1\right] \tag{4}$$

By Equation (4), the seller finds it profitable to include $\theta_i$ into the premium segment whenever the learning buyer virtual type at $\theta_i$ exceeds 1.[8] Hence, among all extensive forms with bottom-up communication and surplus-based rationing, the seller achieves the highest profit by starting the premium segment at the lowest type where the virtual type exceeds 1. Theorem 1 further establishes the seller cannot benefit by employing any other mechanism.

[8]Note that by construction, the learning buyer virtual type is increasing regardless of the distribution, indicating that if it is marginally profitable to include $\theta_i$ in the premium segment, it is also profitable to include $\theta_{i+1}$.

**Theorem 1.** There exists an optimal mechanism $\langle H^S, \beta \rangle$, such that the extensive form $H^S$ features bottom-up communication of length $K$, where $K$ is determined from:

$$K = \min\{j : \gamma^l(\theta_j) \geq 1\} - 1$$

Moreover, the final offers satisfy surplus-based rationing. The buyer terminates communication after getting the first final signal and accepts the offer.

I review the theorem's proof in Section 5.3. I now discuss the role of the learning buyer's virtual type and compare it to its standard counterpart. It is well known that when the buyer has perfect private information about his type, a posted price mechanism is optimal. In this mechanism, the seller similarly divides the buyer types into a rationing segment and a premium segment. However, the degree of rationing is more extreme, as only zero-quality offers are made outside of the premium segment. Under regularity assumption, an optimal posted price (or segmentation) can be obtained by finding the type $\theta_i$, whose *(normalized) informed buyer* virtual type

$$\gamma^{fi}(\theta_i) = \frac{(\theta_i - c)\mu_0(\theta_i)}{(\theta_{i+1} - \theta_i)\Pr(\theta > \theta_i)}$$

exceeds 1.[9] As highlighted in Theorem 1, the learning buyer virtual type serves the same purpose of determining the optimal segmentation. Now, I discuss the distinctions between the two virtual types. For an informed buyer, the numerator of the virtual type represents the profit gain from adding $\theta_i$ into the premium segment, while the denominator summarizes the pressure on the local incentive constraint. When we transition to a learning buyer analog, both of these effects must be captured differently. With a learning buyer, including $\theta_i$ into the premium segment leads to profit gains not only from $\theta_i$ itself but also from all lower types (as the premium segment's consumer surplus increases). Respectively, the learning buyer virtual type for $\theta_i$ accumulates the terms for all preceding types. Moreover, the incentive constraint pressure is also different in my setting. Due to the lack of information, the local incentive constraint does not concern an immediately succeeding type $\theta_{i+1}$ but instead an average type above $\theta_i$. In the next section, I explore how these

[9]The regularity assumption requires $\gamma^{fi}(\cdot)$ is increasing.

differences between the two virtual types manifest themselves in the outcomes of the two models.

## 5.2 Comparison to Posted Price and Welfare Implications

In this section, I compare the outcome allocation in an optimal learning buyer mechanism to that of an optimal posted price mechanism. In what follows, I assume the regularity assumption holds for the informed buyer setting. First, I verify that on an extensive margin, the market expands.

**Corollary 1.** *If a buyer's type $\theta$ is served under the optimal posted-price mechanism, then he is served with a strictly positive quality under the Theorem 1 mechanism. Moreover, if the optimal posted price mechanism is not efficient and does not serve the highest type only, the optimal mechanism in Theorem 1 serves strictly more types.*

*Proof.* First, note that in the learning buyer environment, if the seller can only serve the highest type, it must be that :

$$\gamma^l(\theta_{n-1}) = \gamma^l(\theta_{n-2}) + \frac{\max\{\theta_{n-1} - c, 0\}}{(\theta_n - \theta_{n-1})\mu_0(\theta_n)} < 1$$

But then, it is optimal to only serve the highest type in a posted price mechanism, as $\gamma^{fi}(\theta_{n-1}) < 1$. The market cannot shrink on an extensive margin in this case. Alternatively, if the premium segment includes as least two types, surplus-based rationing ensures that quality remains strictly positive for all buyer types above marginal costs. Furthermore, if the posted price mechanism is not efficient, the optimal mechanism serves strictly more types than the posted price mechanism. □

Now, let's delve into the analysis of the intensive margin. Specifically, I compare the size of the premium segment between the two mechanisms. Suppose the seller charges some price $p$ in the premium segment when using a bottom-up communication with surplus-based rationing. From Figure 4b in Example 1.3, this mechanism not only captures the same profit as a posted price mechanism with $p$, but also reclaims certain surplus that would otherwise be a deadweight loss (DWL) in a posted price mechanism set at price $p$. The extent to which this deadweight loss is captured depends on the degree of rationing, which in turn is driven by the consumer surplus

(CS) generated in the premium segment. When the DWL under a posted price $p$ is relatively small compared to the CS, it indicates that incentives are inexpensive, and the profit potential in the rationing segment is limited. In such cases, the seller can benefit by giving up on some of these incentives by charging a higher premium segment price. In other words, it becomes advantageous for the seller to shift some premium customers into the rationing segment. To formalize this intuition, the following simple corollary provides a lower boundary on the optimal premium segment price in the learning buyer setup.

**Corollary 2.** *Denote $CS(\theta_i)$ and $DWL(\theta_i)$ to be consumer surplus and deadweight loss generated by a posted price $\theta_i$. If $CS(\theta_i) \geq DWL(\theta_i)$, the premium segment price in the mechanism of Theorem 1 is at least $\theta_i$. Moreover, unless the posted price $\theta_i$ is efficient, the optimal premium segment price is strictly higher than $\theta_i$.*

*Proof.* Consider a learning buyer's virtual type evaluated at a price $\theta_i$:

$$\gamma^l(\theta_o) = \sum_{j=1}^{i} \frac{\max\{\theta_j - c, 0\}}{\mathbb{E}_\theta[(\theta - \theta_j)_+]}\mu_0(\theta_j) \leq \sum_{j=1}^{i} \frac{\max\{\theta_j - c, 0\}\mu_0(\theta_j)}{\mathbb{E}_\theta[(\theta - \theta_i)_+]} \leq 1$$

And the inequality is strict if there exists some $\theta_j \in (c, \theta_i)$. □

Therefore, if the deadweight loss at the optimal posted price is not greater than the consumer surplus, the premium segment contracts in the optimal mechanism described in Theorem 1. As a result, consumers are worse off with personal selling compared to a posted price mechanism with no communication. However, it is also crucial to note that personal selling does not necessarily lead to a decrease in consumer surplus. For completeness, I present an example where the premium segment expands instead.

**Example 2.** Suppose the seller bears zero marginal costs, and there are four buyer types. With probability $1/2+\delta$ (for some small $\delta$), the buyer's type is 4, and with a complementary probability, the type is drawn at random from $\{1, 2, 3\}$. The unique optimal posted price is 4. With a learning buyer, the virtual type at 3 is:

$$\gamma^l(3) > \frac{(1/2-\delta)}{3}\frac{2}{2\cdot(1/2+\delta)+(1/2-\delta)/3} + \frac{(1/2-\delta)}{3}\frac{3}{1/2+\delta} \approx 1.286$$

meaning the seller at least includes type 3 into the premium segment when the buyer learns sequentially.

The example hints that we should expect the premium segment to expand when there is not enough variance in the buyer types served under the optimal posted price. Indeed, in this scenario, the generated consumer surplus (and hence incentives for staying in communication) must be too low. The seller would rather lower the premium price to extract higher profit from the rationing segment.

### 5.3 Theorem 1 Proof Overview

In this section, I explain the proof approach of Theorem 1. In particular, I show that when choosing a buyer's strategy $\beta$, the seller should only care about a subset of possible deviating strategies. First, I require that the buyer is *Individually Rational* (IR), meaning that on the path of play, the buyer only makes a purchase whenever he believes the offer is (weakly) preferable to the outside option. Second, I require that the buyer never wants to mimic the reports made by $\theta'$ that he finds out to be lower than his true type. I call the deviations of such form *Myopic Downward Deviations*. I next show that in the corresponding seller's relaxed problem, bottom-up communication with surplus-based rationing is optimal.

**Definition 4.** Say that buyer's strategy $\beta$ is *Individually Rational* (IR) if the buyer type $\theta$ at the outcome of his strategy $\omega^{\beta}(\theta')$ makes (does not make) a purchase when his expected utility from an outcome allocation is strictly positive (negative)

$$\beta(\omega^{\beta}(\theta)) = 1(0) \text{ if } \mathbb{E}[v(\theta', \mathbf{q}^{H_;^S\beta}(\theta), \mathbf{p}^{H_;^S\beta}(\theta))|\theta' \in \hat{\Theta}(\omega^{H_;^S\beta}(\theta))] > (<)0 \tag{IR}$$

Next, for a given buyer strategy $\beta$, I consider *myopic downward deviations* , which take the following form. The buyer follows the specified strategy $\beta$ until reaching some communication stage history $h_l^B \in H_C^B$ with the last signal being positive, $s_l = a$. Starting from this history and for all its successors, the buyer follows the strategy $\beta$ as if he observes the signal outcomes generated by type $\theta'$. I assume that $\theta'$ is such that it gets ruled only by the last signal in the history $h_l^B$: $\theta' \in \hat{\Theta}(h_{l-1}^B) \setminus \hat{\Theta}(h_l^B)$ for $h_{l-1}^B \subset h_l^B$. Formally, for every $(\boldsymbol{\tau}_k, \mathbf{m}_k, \mathbf{s}_k) \supseteq h_l^B$, the buyer plays $\beta((\boldsymbol{\tau}_k, \mathbf{m}_k, \sigma(\theta'|\boldsymbol{\tau}_k)))$. Finally, upon reaching the selling stage history, the buyer makes a purchase if the signal about the offer is positive.

The deviation is myopic because all the types in $\hat{\Theta}(h_l^B)$ ignore any (potentially useful) information after they start deviating. The deviation towards $\theta'$ is downward because this type is ruled out by an "above" signal, meaning the deviating buyer is sure his true type is above $\theta'$. As a result of the deviation, all the deviating types receive an outcome allocation of $\theta'$ — $(\mathbf{q}^{H^S_{,}\beta}(\theta'), \mathbf{p}^{H^S_{,}\beta}(\theta'))$, and a payoff:

$$\sum_{\theta \in \hat{\Theta}(h_l^B)} \max\left\{ v\left(\theta, \mathbf{q}^{H^S_{,}\beta}(\theta'), \mathbf{p}^{H^S_{,}\beta}(\theta')\right), 0\right\} \mu_0(\theta)$$

Downward Incentive Compatibility (D-IC) constraint ensures that none of the deviations of this form are profitable.

**Definition 5** (Downward Incentive Compatibility)**.** Say that a mechanism $\langle H^S, \beta\rangle$ is *Downward Incentive Compatible* (D-IC) if at any communication stage histories $h_l^B \in H_C^B$ with a positive last signal realization ($s_l = a$), there is no profitable myopic downward deviation towards any type $\theta'$ ruled out at $h_l^B$. That is, for all $\theta' \in \hat{\Theta}(h_{l-1}^B) \setminus \hat{\Theta}(h_l^B)$:

$$\sum_{\theta \in \hat{\Theta}(h_l^B)} v\left(\theta, \mathbf{q}^{H^S_{,}\beta}(\theta), \mathbf{p}^{H^S_{,}\beta}(\theta)\right) \cdot \beta\left(\omega^{H^S,\beta}(\theta)\right) \cdot \mu_0(\theta) \geq$$
$$\sum_{\theta \in \hat{\Theta}(h_l^B)} \max\left\{ v\left(\theta, \mathbf{q}^{H^S_{,}\beta}(\theta'), \mathbf{p}^{H^S_{,}\beta}(\theta')\right), 0\right\} \mu_0(\theta) \tag{D-IC}$$

Define the seller's relaxed problem as the one that only requires a mechanism to be IR and D-IC:

**Relaxed Seller's Problem:**

$$\max_{\substack{H^S \in \mathcal{H}, \\ \beta \in \mathcal{B}_{H^S}}} \sum_{\theta \in \Theta} \left(\mathbf{q}^{H^S_{,}\beta}(\theta) - c \cdot \mathbf{p}^{H^S_{,}\beta}(\theta)\right) \mu_0(\theta)$$

subject to (D-IC)

and (IR)

In the next section, I solve this relaxed problem. The solution proceeds in two steps. First, I confirm that it is without loss of optimality to consider bottom-up communication only. Second, I reduce the seller's relaxed problem to a static one and verify it has the same outcome as the optimal surplus-based rationing.

### 5.3.1 Optimal Communication

**Lemma 1** (Bottom-Up Communication)**.** *In the Relaxed Seller's Problem, it is without loss of optimality to consider:*

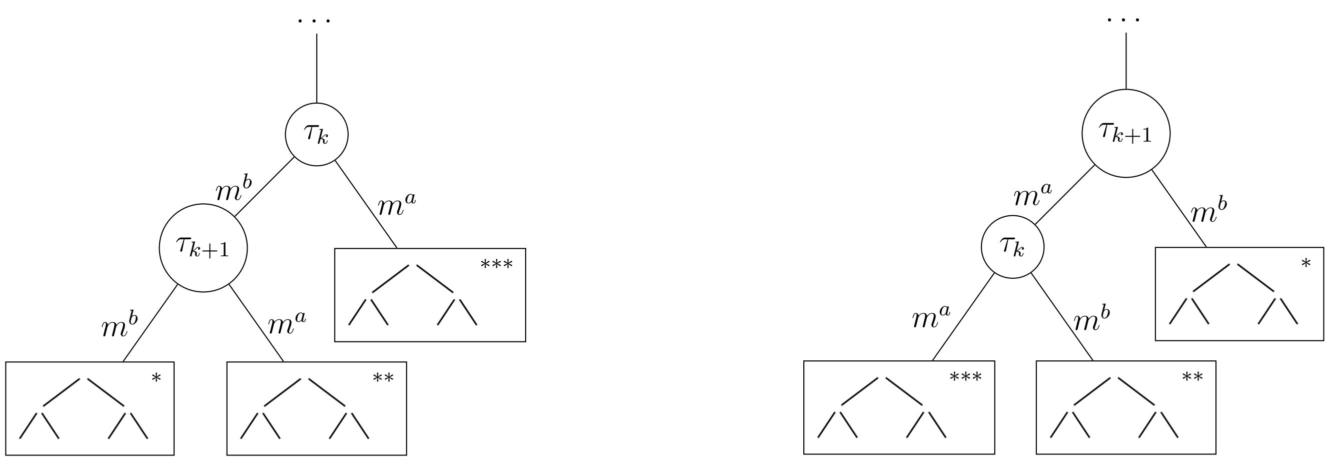
*(i) extensive forms $H^S$ featuring bottom-up communication of length $n-1$*

*(ii) truthful buyer strategy at every buyer plausible history: $\beta(h_k^B) = m^{s_k}$*

*Sketch of proof:* I delegate the formal proof to Appendix A. For the sketch of proof, consider some mechanism $\langle H^S, \beta \rangle$, and suppose that $h_k^S, h_{k+1}^S$ are two successive histories, where the seller first communicates about a higher threshold: $\tau_{k+1} < \tau_k$. An auxiliary Lemma 3 establishes that either $\tau_{k+1}$ is useless and can be substituted with $\tau_k$, or is on the path of play of the buyer types between the two thresholds $(\tau_{k+}, \tau_k]$.

In the latter case, the original mechanism can be replaced with another one $\langle \tilde{H}^S, \tilde{\beta} \rangle$, where the new extensive form $\tilde{H}^S$, reverses the order of the thresholds and appropriately shifts continuation games. Consider Figure 5, and assume the buyer reports truthfully in the original game for simplicity.[10] Then, in the original game the buyer types below $\tau_{k+1}$ proceed to a continuation game $^*$, those in $(\tau_{k+1}, \tau_k]$ — to $^{**}$, and buyer types above $\tau_k$ proceed to continuation game $^{***}$.

[10]For the proof, it is most important that the buyer with a negative signal about $\tau_k$ reports $m^b$ in the original mechanism.




*(a)* $H^S$ *(b)* $\tilde{H}^S$

*Figure 5: Reversing Communication Order*

Consider now an altered extensive form $\tilde{H}^S$ on the right. If the buyer reports truthfully, he gets the same information about his type and proceeds to the same continuation games, respectively, implying the two mechanisms lead to the same outcome allocations. Consider a plausible buyer history $h^B_{k-1}$, which is consistent with a positive signal about $\tau_k$. If the first mechanism is feasible, then there is no profitable myopic downward deviation after a positive signal about $\tau_k$ at any on-path plausible buyer history towards the types that are consistent with prior information, implying that for every $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_k)$:

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_k, \infty)} \max\left\{v\left(\theta, \mathbf{q}^{H^S,\beta}(\theta), \mathbf{p}^{H^S,\beta}(\theta)\right), 0\right\} \mu_0(\theta) \geq$$
$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1})(\tau_k, \infty)} \max\left\{v\left(\theta, \mathbf{q}^{H^S,\beta}(\theta'), \mathbf{p}^{H^S,\beta}(\theta')\right), 0\right\} \mu_0(\theta) \tag{5}$$

As noted above, the threshold $\tau_{k+1}$ in on the path of play for the buyer types who observe a negative signal about $\tau_k$: $\hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_k]$. If the initial mechanism is D-IC, then the deviation after a positive signal about $\tau_{k+1}$ towards any type $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_{k+1})$ must also be unprofitable:

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \tau_k]} \max\left\{v\left(\theta, \mathbf{q}^{H^S,\beta}(\theta), \mathbf{p}^{H^S,\beta}(\theta)\right), 0\right\} \mu_0(\theta) \geq$$
$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \tau_k]} \max\left\{v\left(\theta, \mathbf{q}^{H^S,\beta}(\theta'), \mathbf{p}^{H^S,\beta}(\theta')\right), 0\right\} \mu_0(\theta) \tag{6}$$

Summing up inequalities (5), (6) for any $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_{k+1})$, we get:

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \infty)} \max \left\{ v\left(\theta, \mathbf{q}^{H^S_;\beta}(\theta), \mathbf{p}^{H^S_;\beta}(\theta)\right), 0 \right\} \mu_0(\theta) \geq$$
$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \infty)} \max \left\{ v\left(\theta, \mathbf{q}^{H^S_;\beta}(\theta'), \mathbf{p}^{H^S_;\beta}(\theta')\right), 0 \right\} \mu_0(\theta)$$

which ensures there are no myopic downward deviations when the buyer just gets a positive signal about $\tau_{k+1}$. After this signal, the buyer believes his type is in $\hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \infty)$, so an additional positive signal about $\tau_k$ now rules out any type $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \tau_k]$. From (5), there is no profitable myopic downward deviation after a positive signal about $\tau_k$ in the new extensive form. Hence, if the original mechanism is feasible, so is the new one. Moreover, as it has the same outcome allocation, the seller is indifferent between the two.

By the above, the seller should (without loss) communicate about thresholds in increasing order. Hence, future communication provides no new information once the buyer gets a negative signal. The seller can as well offer such a buyer to end communication immediately. I prove this result formally in Lemma 5.

□

### 5.3.2 Optimal Offers

In what follows, I restrict attention to the mechanisms as in Lemma 1. To characterize an optimal solution to the seller's relaxed problem, it only remains to characterize the offers and purchasing decisions for the selling stages. In the bottom-up mechanism with truthful reports, each threshold rules out the lowest type among those remaining in communication. D-IC constraints reduce to:

$$\sum_{\theta > \theta_i} \left[ \{\max\{v(\theta, \mathbf{q}(\theta), \mathbf{p}(\theta)), 0\} - \max\{v(\theta, \mathbf{q}(\theta_i), \mathbf{p}(\theta_i)), 0\} \right] \mu_0(\theta) \geq 0 \tag{D-IC$_i$}$$

Note that it is without loss of optimality to restrict attention to the offers, which gets accepted by the leaving buyer type: $v(\theta_i, \mathbf{q}(\theta_i), \mathbf{p}(\theta_i)) \geq 0$, as it is always feasible to offer a zero offer.[11] It

[11] This only makes D-IC$_i$ slacker with no impact on seller's profit.

is more convenient to reformulate the choice of allocation in terms of the buyer's indirect utility $U(\theta) \equiv \theta\mathbf{q}(\theta) - \mathbf{p}(\theta)$. Finally, the seller's relaxed problem gets reduced to the following static one:

**Equivalent Static Problem:**

$$\max_{\mathbf{q}:\Theta\to[0,1],U:\Theta\to\mathbb{R}} \quad \sum_{i=1}^{n} \mu_0(\theta_i)\left[(\theta_i - c)\mathbf{q}(\theta_i) - U(\theta_i)\right]$$

subject to

$$\sum_{j>i}^{n} U(\theta_j)\mu_0(\theta_j) \geq \sum_{j>i}^{n} \left[U(\theta_i) + \mathbf{q}(\theta_i)(\theta_j - \theta_i)\right]\mu_0(\theta_j) \tag{D-IC$_i$}$$

$$U(\theta_i) \geq 0 \tag{IR$_i$}$$

I now solve this problem and verify that surplus-based rationing is optimal if a learning buyer virtual type determines the premium segment. First, when the quality of type $\theta_i$ is interior, the respective D-IC$_i$ must bind. In this case, the seller wants to make $U(\theta_i)$ as low as possible (as allowed by IR$_i$) to slacken the binding incentive constraint. Otherwise, the seller could proportionally redistribute this utility among the higher types, keeping the continuation value of communication the same (Lemma 9). The next lemma ensures that there must be a unique threshold type determining which buyer types receive the premium quality in the optimum.

**Lemma 2.** *Suppose that $\mathbf{q}^*(\cdot), U^*(\cdot)$ is a solution to the Equivalent Static Problem. Then, a unique threshold type $\tilde{\theta} \leq \theta_n$ exists, determining the premium segment. Namely, $\tilde{\theta}$ is such that $\mathbf{q}^*(\theta) < 1$ for all $\theta < \tilde{\theta}$ and $\mathbf{q}^*(\theta) = 1$ for all $\theta \geq \tilde{\theta}$.*

The proof of the lemma is delegated to Appendix B. I now combine the insights from Lemma 9 and Lemma 2. If $\mathbf{q}^*(\cdot), U^*(\cdot)$ is a solution to the problem, then there is some threshold type that determines whether the buyer gets a premium quality. Whenever a buyer type gets a (positive) rationed quality, his surplus is fully extracted. Consequently, the quality of every buyer type $\theta_i$ belonging to the rationing segment is determined from D-IC$_i$ as follows:

$$\mathbf{q}(\theta_i) = CS/\left(\sum_{j>i}^{n}\left[(\theta_j - \theta_i)\right]\mu_0(\theta_j)\right) \tag{7}$$

where $CS$ is the total consumer surplus in the mechanism. Now, suppose that the seller uses type $\theta_k$ as a threshold type for the segmentation and consider the effect of shifting an additional type $\theta_{k-1}$ to the premium segment. To satisfy D-IC$_{k-1}$, the shift must lead to a positive change in consumer surplus. Denote this change as $\Delta CS$. Then by Equation (7) the qualities in the rationing segment all increase by $\mathbf{\Delta q}(\theta_i) = \Delta CS / \left( \sum_{j>i}^{n} [(\theta_j - \theta_i)]\, \mu_0(\theta_j) \right)$. The total change in the seller's payoff is

$$\underbrace{\sum_{i=1}^{k-1} \max\{\theta_i - c, 0\} \mathbf{\Delta q}(\theta_i) \mu_0(\theta_i)}_{\text{Change in Total Surplus}} - \Delta CS = \Delta CS \cdot (\gamma^l(\theta_{k-1}) - 1) \tag{8}$$

Hence, there is no profitable segmentation change whenever the threshold type is $\theta_k = \min\{\theta_i : \gamma^l(\theta_i) \geq 1\}$.

Now suppose that the segmentation at $\theta_k$ is optimal. If the seller attempts to change consumer surplus while keeping the segmentation the same, the change in the payoff is again given by Equation (8), and it is negative if the segmentation is optimal, as $\gamma^l(\theta_{k-1}) < 1$. The minimal consumer surplus $\underline{CS}$ that is incentive-compatible with the segmentation at $\theta_k$ is determined by D-IC$_k$: $\underline{CS} = \sum_{j>k}^{N} [(\theta_j - \theta_k)]\, \mu_0(\theta_j)$. In the Equivalent Static Problem, the seller is indifferent between different ways of generating this consumer surplus as long as it is feasible. In particular, this consumer surplus can be generated by charging a constant price $\theta_k$ at the premium segment.

Note that bottom-up communication of length $k-1$ with the surplus-based rationing generates the same outcome as the optimal solution to the relaxed static problem. Then, the mechanism described in Theorem 1 solves Seller's relaxed problem while feasible in the initial one. This concludes the proof of Theorem 1.

# 6 Discussion

In this section, I discuss the main assumptions used in the paper, focusing on the interpretation of the learning technology.

*No Instant Learning of the Reservation Price.* At the core of the learning buyer setting is the assumption that the buyer cannot accurately determine his reservation price for any product quality, even if he observes a product of that quality. This assumption is crucial to maintain the learning process, as it prevents the buyer from immediately understanding his type after the first interaction with a product. Specifically, the assumption implies that if the buyer gets presented with a menu $\{(q_1, p_1); (q_2, p_2)\}$, the buyer cannot ask himself the questions of the following form:

(i) Would I be willing to purchase quality $q_1$ at some other price $p' \in \mathbb{R}$?

(ii) Would I be willing to pay a premium $\Delta p \neq p_1 - p_2 \in \mathbb{R}$ for a quality upgrade $\Delta q = q_1 - q_2$?

In support of this assumption, experimental studies have shown that decision-makers often make errors when estimating their willingness to pay for a product, exhibiting phenomena like the endowment effect (Kahneman, Knetsch & Thaler (1991)) and the anchoring effect (Tversky & Kahneman (1974)). These errors seem to diminish as the buyer gains more experience with similar decision problems (List (2003), Bateman et al. (2008), Plott & Zeiler (2005)). The discovered preference hypothesis explains these observations, suggesting that decision-makers need to discover their preferences through experience. This theory aligns with how I model consumers' learning in this paper.

*Role of Samples.* I now discuss the role of the samples used for the implementation of information disclosure. In describing the optimal solution, I remained agnostic about the exact samples the seller may present to the buyer to achieve the desired learning. In particular, the solution in Theorem 1 can be achieved by directly exposing the buyer to the respective selling stage offer in the same period. This interpretation of the samples is in line with the seller guiding the buyer through the products in her store or explaining the options available in the catalog. To illustrate, consider the mechanism described Example 1.4, which can be implemented as follows. At each period $t$, the seller shows the buyer a product $(q_t, p_t)$, which the buyer compares to the outside option $(0, 0)$. After getting a signal about such a menu, the buyer decides whether to proceed to the selling stage with a final offer $(q_t, p_t)$ or stay in communication. The concrete evidence assumption ensures that using final offers for information disclosure is incentive-compatible. However, this implementation would break down under alternative (very reasonable) assumptions about the buyer's store experience. For instance, the implementation would not be feasible if the buyer could learn from any menu composed of previously shown products.

To demonstrate, I revisit the mechanism described in Example 1.4 and consider a specific period, say 0.25. Suppose that with the use of past offers, a buyer staying in communication can learn if his type is below some $\bar{\theta}$. In this scenario, the continuation value of communication is $(\bar{\theta}-\tilde{\theta})^2/[2(\bar{\theta}-0.25)]$, while the expected value of purchasing $(q_{0.25}, p_{0.25})$ immediately is $q_{0.25}(\bar{\theta}-0.25)/2$. To assess the buyer's optimal decision with this additional information, we look at the ratio between the two values:

$$\frac{1}{q_{0.25}}\frac{(\bar{\theta}-\tilde{\theta})^2}{(\bar{\theta}-0.25)^2}$$

This ratio is increasing in $\bar{\theta}$ and is strictly less than 1 when $\bar{\theta}=0.6$, which is feasible for the buyer to learn about by analyzing the menu $\{(q_{0.25}, p_{0.25}), (q_{0.1}, p_{0.1})\}$. Consequently, after considering this menu, some buyer types above 0.25 would prefer to terminate communication immediately instead of continuing the learning process. This example illustrates how the buyer's ability to consider hypothetical menus could undermine the incentive compatibility of the mechanism when final offers are used as sample products.[12]

Likewise, the implementation with final offers would not be feasible if the buyer could recall and demand any of the previously seen products. If the buyer could choose to purchase any of the presented product variations, he would not select the outcome of the bottom-up mechanism. This is because certain buyer types regret not leaving earlier after learning their type. Therefore, for the relevant applications, my model provides an upper limit of achievable profit, as it represents a very extreme version of the buyer's experience manipulation.

However, if the seller can use different sample products while communicating, the incentive compatibility can be restored even if the buyer can consider all familiar products. For example, suppose at period $i$ of the communication stage, the seller discloses information to the buyer about his preference between a sample product $(q_i^s, p_i^s)$ and an outside option $(0,0)$. The seller can find

[12]Nevertheless, it is also important to note that even if the seller only uses final offers to provide information, and the buyer can use information from all familiar products, the simple single offer mechanism described in Example 1.2 remains feasible. The model does not collapse to the informed buyer set-up.

the samples such that $p_i^s/q_i^s = \theta_i$, so that the binary menu $\{(q_i^s, p_i^s); (0,0)\}$ induces the desired learning of the bottom-up communication, while any menu of the two samples $\{(q_k^s, p_k^s); (q_l^s, p_l^s)\}$ is completely uninformative. One way to achieve this is to use samples with a positive constant quality $q_i^s \equiv q > 0$ and progressively increasing prices $p_i^s = \theta_i q_i^s$, so that any buyer type in $\Theta$ prefers $(q_k^s, p_k^s)$ in a menu $\{(q_k^s, p_k^s); (q_l^s, p_l^s)\}$ as long as $k < l$. Hence, with these samples, no additional useful information is available even when the buyer can additionally consider the menus of all previously seen products. Thus, different assumptions about a buyer's ability to process information can impact the seller differently, depending on how flexible she is when designing sample products for the communication stage.

*Random Communication.* It's also worthwhile to explore the implications of the assumption concerning the seller's use of deterministic strategies. Crucially, if only deterministic communication is allowed, the buyer's ability to understand the quality of the presented products becomes irrelevant. Since the seller commits to a fixed communication schedule, the buyer knows precisely which information is provided at every stage. In contrast, if the seller could randomize between different sample products (thresholds), the buyer may be unsure about the exact content of the signal realization. In this scenario, the seller can extract essentially the whole market surplus. I illustrate the idea with the following simple two-type example.

**Example 3.** Suppose $\Theta = \{\theta_1, \theta_2\}$ and the buyer does not observe a threshold used to generate signal during the communication stage (while this information is available to the seller). Suppose the buyer-seller interaction proceeds as follows.

(i) In period 1, the seller reveals to the buyer if he is above $\theta_1$: $\tau_1 = \theta_1$. The buyer observes the signal generated by $\tau_1$, and decides whether to stay in communication or proceed to the selling stage, where he gets an offer $(1, \theta_2 - \varepsilon)$.

(ii) If the buyer stays, communication continues for another $N$ periods. In each of these periods, the seller randomly draws (with equal probability) one of the two thresholds $\{\theta_1 - \Delta, \theta_1 + \Delta\}$ for $\Delta$ sufficiently small. Let $\tau_k$ be the realized threshold in period $k \in \{2, \dots, N+1\}$, observable to the seller but not to the buyer. At the end of each communication period, the buyer is asked to give feedback about the signal realization $\sigma(\cdot|\tau_k)$. If the buyer reports "correctly" after each period, meaning $m_k = m^{\sigma(\theta_1|\tau_k)}$, the buyer gets to a selling stage with an offer $(1, \theta_1)$. Otherwise, he gets a zero offer $(0, 0)$.

Let us verify that if $N$ is large enough, it is incentive compatible for a buyer observing a positive signal in period 1 (that is, a buyer of type $\theta_2$) to terminate communication immediately. Suppose, by contradiction, such a buyer stays. For any threshold realization in $\{\theta_1 - \Delta, \theta_1 + \Delta\}$, the buyer of type $\theta_2$ observes a signal realization "above" and has a 1/2 chance of guessing $\sigma(\theta_1|\tau_k)$ correctly. Hence, the buyer gets a selling stage offer $(1, \theta_1)$ with at most $1/2^N$ probability, and an offer $(0, 0)$ otherwise. Consequently, the value of staying in communication for a buyer type $\theta_2$ is $(\theta_2 - \theta_1)/2^N$, and can be made arbitrarily small with large $N$.

In other words, the buyer's learning only becomes valuable when it is combined with the seller's information about the threshold. Having commitment power, the seller leverages this complementarity in information to essentially extract the entire surplus whenever the threshold selection is unknown to the buyer. However, if the buyer perfectly understands the characteristics of the sample products (observes the realized thresholds used to generate signals), I conjecture the seller cannot improve the outcome further with random strategies.

# 7 General Utility and Convex Costs

In this section, I explore generalizations of the model and verify that the key insights concerning optimal information disclosure remain applicable even within more complex environments. Suppose the model is as described in Section 2, but allows for more general preferences of both agents. In particular, suppose a buyer of type $\theta$ derives utility $v(\theta, q, p) = u(\theta, q) - p$ when purchasing product $(q, p)$, and the seller bears costs $c(q)$ of producing quality $q \in [0, \bar{Q}]$. I assume $c(\cdot)$ is convex and twice continuously differentiable, and $c(0) = 0$. Additionally, I impose the following assumptions on $u(\cdot)$:

(i) $u(\theta, 0) = 0$ for every $\theta$

(ii) $u(\theta, \cdot)$ is twice continuously differentiable and is concave for every $\theta$

(iii) $u(\cdot,\cdot)$ satisfies increasing differences: $u(\tilde{\theta},\tilde{q})-u(\tilde{\theta},\hat{q}) \geq (>) u(\hat{\theta},\tilde{q})-u(\hat{\theta},\hat{q})$ for every $\tilde{\theta} \geq (>)\hat{\theta}$, $\tilde{q} \geq (>)\hat{q}$

To clarify, part (i) imposes that there is a base quality 0, which is not valued by any type, while part (iii) means that higher types value any increase in quality more than lower types do. It also ensures that the learning technology can be easily accommodated in this generalized setting. As increasing differences imply single crossing, learning about ordinal preferences is equivalent to learning whether a buyer's type exceeds a specific threshold.

Theorem 2 establishes the main insight about optimal communication is preserved in this generalized setting: it is optimal for the seller to reveal information from the bottom-up. To describe the optimal selling stage offers, let me introduce *surplus-based distortion* in quality in analog to surplus-based rationing used in the paper's main section. As before, the degree of distortion depends on the consumer surplus generated in the mechanism. Given the consumer surplus $CS$, the distorted quality for type $\theta_i$, $\mathbf{q}^d(\theta_i, CS)$, is implicitly defined as follows:

$$\mathbb{E}\left[\left(u\left(\theta, \mathbf{q}^d(\theta_i, CS)\right) - u\left(\theta_i, \mathbf{q}^d(\theta_i, CS)\right)\right)_+\right] = CS \tag{9}$$

with a convention $\mathbf{q}^d(\theta_i, CS) = +\infty$, if the solution Equation (9) does not exists. In surplus-based distortion, the seller serves the efficient quality whenever feasible and distorted quality otherwise. Let $\mathbf{q}^e(\theta)$ denote the efficient quality for type $\theta$ buyer:

$$\mathbf{q}^e(\theta_i) \equiv \underset{[0,\bar{Q}]}{\text{Argmax}}\{u(\theta_i, q) - c(q)\}$$

For price determination, the market gets split into two segments. In the lower segment, the seller extracts the whole surplus. In the upper segment, the prices are set by the usual envelope formula from Mussa & Rosen (1978) given the offered qualities are efficient for all buyer types. Let $CS^{MR}(\theta_i)$ denote the expected consumer surplus when this envelope pricing starts from $\theta_i$, and this type gets his full surplus extracted:

$$CS^{MR}(\theta_i) \equiv \sum_{j=i}^{n-1} \Pr(\theta > \theta_j) \left[ u(\theta_{j+1}, \mathbf{q}^e(\theta_j)) - u(\theta_j, \mathbf{q}^e(\theta_j)) \right]$$

To determine the segment to which each type belongs, we need to find the $CS^{MR}(\theta_i)$ that can achieve the desired consumer surplus $CS$ and adjust the price paid by $\theta_i$, if necessary.

**Definition 6.** Given an extensive form $H^S$ with bottom-up communication of length $n-1$, say that the final offers $\{(q_i, p_i)\}_{i=1}^n$ feature surplus-based distortion if there exists a consumer surplus $CS \in \mathbb{R}_+$, such that $q_i = \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}$ and

$$p_i = \begin{cases} u(\theta_i, q_i), \text{ if } i < K(CS) \\ u(\theta_i, q^e(\theta_i)) - \frac{CS - CS^{MR}(\theta_i)}{\Pr(\theta \geq \theta_i)}, \text{ if } i = K(CS), \\ p_{K(CS)} + \sum_{j=K+1}^{i} \left[ u(\theta_j, \mathbf{q}^e(\theta_j)) - u(\theta_j, \mathbf{q}^e(\theta_{j-1})) \right] \text{ if } i > K(CS) \end{cases},$$

where $K(CS) = \min\{i : CS^{MR}(\theta_i) \leq CS\}$

Lemma 8 in Appendix C confirms that in any extensive form $H^S$ with bottom-up communication and surplus-based distortion, a buyer type $\theta_i$ abandons communication after round $i$ in an optimal buyer's strategy. Moreover, Theorem 2 demonstrates that the seller cannot achieve better outcomes than those attainable with mechanisms of this form. In addition, a generalized version of a learning-buyer virtual type determines the optimal consumer surplus used for the construction of the final offer.

**Theorem 2.** There exists an optimal mechanism $\langle H^S, \beta \rangle$, such that the extensive form $H^S$ features bottom-up communication for $n-1$ rounds, where the final offers at every stage satisfy surplus-based distortion. Additionally, if the efficient quality is interior for every buyer type $\mathbf{q}^e(\theta_i) \in (0, \bar{Q})$, then the optimal consumer surplus $CS$ is either zero (and only the highest type is served) or is determined by:

$$\sum_{i=1}^{n-1} \frac{u'_q(\theta_i, q_i)) - c'(q_i)}{\mathbb{E}\left[\left(u'_q(\theta, q_i) - u'_q(\theta_i, q_i)\right)_+\right]} = 1 \tag{10}$$

The proof adopts the same approach as demonstrated for the linear case. Lemma 1 is readily applied to this general case: bottom-up communication is optimal in a relaxed problem that only allows for myopic downward deviations and purchasing deviations. Using bottom-up communication allows me to find an equivalent static problem where the suggested mechanism can be shown to be optimal.

In Corollary 3, I show that provided there is not too much probability weight on the highest type, compared to Mussa & Rosen (1978), the "no distortion at the top" gets expanded and includes a whole range of higher types. Even though the seller still screens different buyer types at the top of the distribution, she does not distort their qualities.

**Corollary 3.** *If the distribution of buyer types is sparse enough, at least two types get served efficient quality in the optimal mechanism of Theorem 2.*

*Proof.* Suppose $\theta_{n-1}$ is not served the efficient quality under consumer surplus $CS$. $\mathbf{q}^d(\theta_{n-1}, CS) < \mathbf{q}^e(\theta_{n-1})$ and consider

$$\frac{u'_q(\theta_{n-1}, q_{n-1})) - c'(q_{n-1})}{\mu_0(\theta_n)\left[\left(u'_q(\theta_n, q_{n-1}) - u'_q(\theta_{n-1}, q_{n-1})\right)_+\right]}$$

For any such consumer surplus, we can find a low enough $\mu_0(\theta_n)$, that the expression above becomes larger than 1. But then $CS$ is not optimal by Theorem 2. $\square$

Lemma 7 also points out that compared to the linear model, in a more general set-up, it is possible that some of the buyer types who receive efficient quality still receive zero utility under an optimal mechanism.

**Example 4.** Suppose $\Theta = [0, 1]$, and the buyer's utility is $v(\theta, q, p) = \theta q - p$. The seller's costs are quadratic $c(q) = q^2/2$. In Figure 6, I plot the quality allocation induced by the optimal mechanism from Theorem 2.

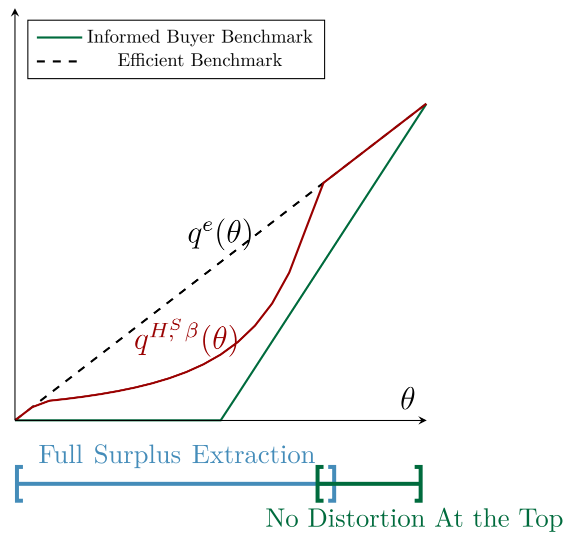



*Figure 6: Induced Allocation of the Optimal Mechanism*

# 8 Horizontal Differentiation

In this section, I consider the most basic extension of the model permitting some horizontal differentiation between products. Let's imagine the seller is a kitchen design firm that offers kitchens made of either stone or wood. Different grades are available for each of the materials. The buyer types differ in how extreme their preferences are about the kitchen materials. During the communication stage, the seller shows binary menus of samples, which feature different materials and grades. As before, the buyer only learns which of the two samples he likes more.

I refer to the horizontal aspect of the product characteristics as its location $l \in L = \{-1, 1\}$ (kitchen material), and to the vertical aspect as quality $q \in [0, 1]$ (material grade). Irrespective of the location, the seller bears a constant marginal cost $c$ for producing vertical quality. The buyer with more extreme preferences towards a product of location $l$ places a higher value on the quality associated with that location. Formally, the buyer of type $\theta \in \Theta \subseteq [-1, 1]$ derives utility $u(\theta, l, q)$ from a product $(l, q)$:

$$u(\theta, l, q) = (\bar{v} - |\theta - l|) \cdot q$$

For the two products of different locations $\{(-1, q_1); (1, q_2)\}$, the buyer learns whether he prefers the first product, which is equivalent to learning whether $\theta \leq (\bar{v}-1)(q_1-q_2)/(q_1+q_2)$. After the communication stage ends, the seller makes a final take-it-or-leave-it offer $(l, q, p)$. If the buyer of type $\theta$ accepts the offer, he gets a payoff $v(\theta, l, q, p) = u(\theta, l, q) - p$.

Assuming $\bar{v}$ is either sufficiently high or does not exceed $1-c$, I verify that the key insights of the model remain applicable in this setting. In particular, the optimal communication unfolds as follows. First, the seller splits the market between two locations by disclosing whether the buyer type $\theta$ is above or below some $\tau_1 \in \Theta$. Given the feedback from the buyer about $\tau_1$, all subsequent offers during the selling stage will be of location $-1$ (in response to "below" reports) or of location 1 (in response to "above" reports). Then, for each of these locations' submarkets, the seller independently bottom-up communicates about the extremeness of the buyer's type $\theta$.

**Proposition 1.** *If $\bar{v}$ satisfies the premise Lemma 10, then in a model of Hotelling differentiation, there exists an optimal mechanism $\langle H^S, \beta \rangle$, such that if the buyer reports "below" ("above") about $\tau_1 = \theta_k$ for some $\theta_k \in \Theta$, in the continuation game the seller:*

*(i) Offers location $-1$ ($1$) in all succeeding selling stage sequences.*

*(ii) Bottom-up communicates about $-\theta$ ($\theta$).*

*(iii) At the corresponding selling stages, presents a quality-price combination that involves surplus-based rationing optimal for a (vertical) type space $\Theta^b \equiv \{\bar{v}-1-\theta_k, \ldots, \bar{v}-1-\theta_1\}$ ($\Theta^a \equiv \{\bar{v}-1+\theta_{k+1}, \ldots, \bar{v}-1+\theta_n\}$).*

The proof of Proposition 1 can be found in Appendix D. I illustrate the solution to the problem in Example 5.

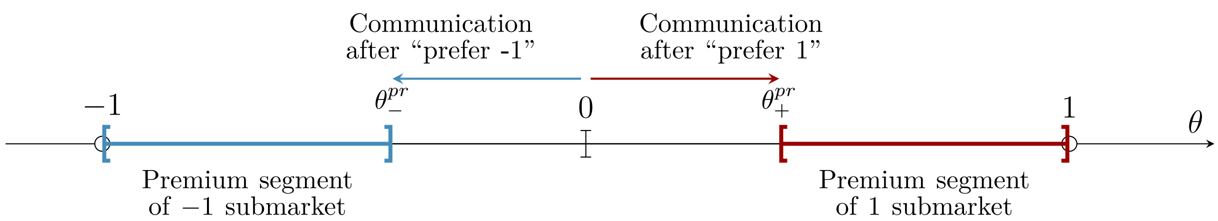
**Example 5.** Assuming $\bar{v} = 3$, $\theta \sim U[-1, 1]$, and the seller has zero marginal cost $c = 0$, the optimal strategy for the seller is to divide the market between the two locations evenly. The initial samples are only different in their locations but have the same quality. If the buyer prefers location $-1$ in this initial sample menu, the seller gradually decreases the threshold by making the trade-off between preferred location and quality more extreme. Given types $[-1, 0]$ are all served $l = -1$, we get a vertically-differentiated type $(\bar{v}-1-\theta) \sim U[2, 3]$, and the optimal premium segment

$[-1, \theta_-^{pr}]$ on $-1$ submarket is determined by:

$$\gamma^l(\bar{v} - 1 - \theta_-^{pr}) = \int_2^{2-\theta_-^{pr}} \frac{x}{(3-x)^2/2} = 1 \Rightarrow \theta_-^{pr} \approx -0.192$$

The premium market $[\theta_+^{pr}, 1]$ on submarket served $l = 1$ is derived symmetrically.

*Figure 7: Optimal Communication in Hotelling Differentiation*




# 9 Conclusion

In this paper, I examine a model of personal selling: the seller strategically designs a salesforce manual, detailing the provision of new information and final product offers based on the interaction with the buyer. The seller can select sample products to showcase to the buyer. These observed sample products aid the consumer in evaluating the trade-offs between distinct product attributes. Considering the private nature of the buyer's learning, misrepresentation to the seller is possible. I establish that the seller's primary concern should be countering myopic downward deviations, where the buyer falsely portrays themselves as a lower type, which is ruled out by the most recent signal. To mitigate such deviations, the seller should gradually disclose the buyer's type, starting from the bottom. This approach enables the seller to more effectively screen different buyer types, particularly the lower ones, as sequential information disclosure pools the incentives of yet uninformed buyer types.

The model yields new insights into how a seller can strategically leverage a buyer's learning to their advantage, particularly when the seller possesses substantial control over the buyer's product experience. Future research is indeed to understand how alternative assumptions could affect the optimal salespeople's techniques.

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# Appendices

## A Proofs for Section 5.3.1

For convenience, in the following proofs, I use the following notation. For every vector, $\mathbf{a}_k \in \mathbb{R}^k$, let $\mathbf{a}_{[l,k]}$ denote the elements of $\mathbf{a}_k$ from $k$ to $l$.

**Lemma 3** (No-Useless Information)**.** *In the relaxed problem, it is without loss to consider $\beta$ and $H^S$, such that if a successive history $h^S_{k+1} \supset h^S_k$ is not on path of a buyer who observes a negative signal about $\tau_k$ and a positive signal about any past lower thresholds, then $\tau_{k+1} \geq \tau_k$.*[13]

*Proof.* Suppose that there are histories $h^S_k = (\boldsymbol{\tau}_k; \mathbf{m}_{k-1})$ and $h^S_{k+1} = (\boldsymbol{\tau}_{k+1}; \mathbf{m}_k)$ in $H^S$, such that $\tau_{k+1} \leq \tau_k$ and it is not on-path of a buyer who observes a negative signal about $\tau_k$: $\beta(h^B_k) \neq m_k$ for any plausible buyer history induced by $h^S_k$: $h^B_k = (h^S_k; \mathbf{s}_{k-1}, b)$. Then, the seller can employ a new mechanism $\langle \tilde{H}^S, \tilde{\beta} \rangle$, which substitutes $\tau_{k+1}$ with $\tau_k$ at $h^S_{k+1}$ and all succeeding histories. I describe the construction formally below.

First, if a history $h^S$ is not $h^S_{k+1}$ or does not succeed $h^S_{k+1}$, it is included in $\tilde{H}^S$. Similarly, if a plausible buyer's history is included in both $\tilde{H}^S, H^S$ the buyer plays the same action: $\tilde{\beta}(h^B) = \beta(h^B), \forall h^B \in \tilde{H}^S \cap H^S$.

Then, I replace $h^S_{k+1}$ with $\tilde{h}^S_{k+1} = (\boldsymbol{\tau}_k, \tau_k; \mathbf{m}_k)$, and similarly every succeeding histories $h^S_l = (\boldsymbol{\tau}_l; \mathbf{m}_{l-1})$ gets replaced by $\tilde{h}^S_l = (\boldsymbol{\tau}_k, \tau_k, \boldsymbol{\tau}_{[k+1,l]}; \mathbf{m}_{l-1})$. For a plausible buyer's history $\tilde{h}^B_l = (\tilde{h}^S_l; \mathbf{s}_l)$, which has a positive signal about $\tau_k$ ($s_k = a$), one can find a buyer plausible history in the original game with coinciding signal realizations $h^B_l = (h^S_l; \mathbf{s}_l)$. [14] Then, $\tilde{\beta}$ is defined so that the buyer plays the same action under $\tilde{\beta}$ at $\tilde{h}^B_l$, as under the original strategy $\beta$ at $h^B_l$: $\tilde{\beta}(\tilde{h}^B_l) = \beta(h^B_l)$. The construction of new selling stage histories and strategies at these histories is analogous. By construction, the new mechanism is (IR) as long as $\langle H^S, \beta \rangle$ is. Note also that when defined this way, the new mechanism $\langle \tilde{H}^S, \tilde{\beta} \rangle$ has the same outcome allocation as the original one. Moreover, since both $\tau_k$ or $\tau_{k+1}$ provide no useful information after a positive signal about $\tau_k$, $\hat{\Theta}(h^B_l) = \hat{\Theta}(\tilde{h}^B_l)$. But then, if the initial mechanism is (D-IC), so is the new one.

[13]In other words, not on the path of play of $\theta \in (\max\{\tau_j : \tau_j < \tau_k\}, \tau_k]$.

[14]Note that if the buyer observes a negative signal about $\tau_k$, $\tilde{h}^S_{k+1}$ is not the path of play. So I can define $\tilde{\beta}$ arbitrarily for histories buyer plausible $(\tilde{h}^S_l; \mathbf{s}_l)$ with $s_k = b$ with no impact on constraints or the seller's payoff.

□

**Lemma 4** (Increasing Thresholds)**.** *For the relaxed problem, it is without loss of optimality to consider extensive forms $H^S$, which communicate about thresholds in increasing order: if $h_k^S \subseteq h_{k+1}^S \in H_C^S$, then $\tau_{k+1} \geq \tau_k$.*

*Proof.* Lemma 4 Take some extensive form $H^S$ and buyer strategy $\beta \in \mathcal{B}_{H^S}$, which satisfy D-IC and IR. Suppose by a way of contradiction that there are histories $h_k^S = (\boldsymbol{\tau}_k; \mathbf{m}_{k-1})$ and $h_{k+1}^S = (\boldsymbol{\tau}_k, \tau_{k+1}; \mathbf{m}_k)$ in $H^S$, such that $\tau_{k+1} < \tau_k$.

If $h_{k+1}^S$ is not on-path of the buyer type who gets a negative signal about $\tau_k$, apply Lemma 3. Otherwise, I consider a new extensive form $\tilde{H}^S$, which effectively changes the order of thresholds while preserving the relevant choice over available information and final offers. Similarly, I construct a new buyer strategy $\tilde{\beta} \in \mathcal{B}_{\tilde{H}^S}$ so that the buyer gets the same allocation and signal realizations in the outcome of the new extensive form. I describe the construction formally below, dividing it into three steps.

**Step 1**. First, let $\tilde{H}^S$ contain all the same histories as the initial extensive form $H^S$ that are not subhistories of $h_k^S$. For all these coinciding histories, let the buyer play the same strategy: $\tilde{\beta}(h^B) = \beta(h^B)$, whenever $h^B \in H^B \cap \tilde{H}^B$. Clearly, if $\beta$ satisfies D-IC, there are no profitable downward deviations at each of these buyer's histories.

**Step 2**. Now I alter the order of thresholds, starting from $h_k^S$ and $h_{k+1}^S$, and in all successive histories. First, for the new extensive form $\tilde{H}^S$, I replace histories $h_k^S$ and $h_{k+1}^S$ with histories $\tilde{h}_k^S = (\boldsymbol{\tau}_{k-1}, \tau_{k+1}; \mathbf{m}_{k-1})$ and $\tilde{h}_{k+1}^S = (\boldsymbol{\tau}_{k-1}, \tau_{k+1}, \tau_k; \mathbf{m}_{k-1}, \mathbf{m}_{k-1}, m^a)$, respectively. I now alter the buyer's strategy $\tilde{\beta}$ at the induced plausible buyer's histories by making $\tilde{\beta}$ truthful at $\tilde{h}_k^S$ and $\tilde{h}_{k+1}^S$: $\tilde{\beta}((\tilde{h}_k^S; \mathbf{s}_{k-1}, s)) = m^s$ and $\tilde{\beta}((\tilde{h}_{k+1}^S; \mathbf{s}_k, s)) = m^s$. Now I verify that there are no profitable deviations at any on-path buyer plausible histories induced by these replacements. Since I have not yet defined the mechanism fully, let me assume for now that the new mechanism is IR and has the same outcome allocation. I later verify this assumption is true.

**Assumption 1.** Assume that the new mechanism $\langle \tilde{H}^S, \tilde{\beta} \rangle$ is (IR) and has the same outcome allocation as the original mechanism $\langle H^S, \beta \rangle$.

If Assumption 1 is true, then there are no profitable deviations at on-path plausible buyer histories induced by the replaced histories: $\tilde{h}^B_k = (\tilde{h}^S_k; \mathbf{s}_{k-1}, a)$ or $\tilde{h}^B_{k+1} = (\tilde{h}^S_{k+1}; \mathbf{s}_{k-1}, a, a)$. Let me prove this statement. Consider some on-path plausible buyer history: $h^B_{k-1} = (\boldsymbol{\tau}_{k-1}, \mathbf{m}_{k-2}, \mathbf{s}_{k-1})$, where $(\boldsymbol{\tau}_{k-1}, \mathbf{m}_{k-2})$ is a subhistory of $h^S_k$ that I attempt to replace. By the premise of the Lemma, $\langle H^S, \beta \rangle$ satisfies (D-IC) and (IR). Then, there is no profitable myopic downward deviation at a succeeding history $h^B_k = (\boldsymbol{\tau}_k; \mathbf{m}_{k-1}; \mathbf{s}_{k-1}, a)$, where the buyer learns additionally his type is higher than $\tau_k$, towards any type $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_k]$ that is rule out by this additional signal.

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_k, \infty)} \left[ \max \left\{ v\left(\theta, \mathbf{q}^{H^S_i \beta}(\theta), \mathbf{p}^{H^S_i \beta}(\theta)\right), 0 \right\} - \max \left\{ v\left(\theta, \mathbf{q}^{H^S_i \beta}(\theta'), \mathbf{p}^{H^S_i \beta}(\theta')\right), 0 \right\} \right] \mu_0(\theta) \geq 0 \tag{11}$$

Similarly, there is no profitable myopic downward deviation at $h^B_{k+1} = (\boldsymbol{\tau}_{k+1}; \mathbf{m}_k; \mathbf{s}_{k-1}, b, a)$, where the buyer first learns after $h^B_{k-1}$ his type is below $\tau_k$ but above $\tau_{k+1}$ towards any $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_{k+1}]$ which is just ruled out by the last signal:

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \tau_k]} \left[ \max \left\{ v\left(\theta, \mathbf{q}^{H^S_i \beta}(\theta), \mathbf{p}^{H^S_i \beta}(\theta)\right), 0 \right\} - \max \left\{ v\left(\theta, \mathbf{q}^{H^S_i \beta}(\theta'), \mathbf{p}^{H^S_i \beta}(\theta')\right), 0 \right\} \right] \mu_0(\theta) \geq 0 \tag{12}$$

By Assumption 1, the outcome allocation of $\tilde{\beta}$ in the new extensive form $\tilde{H}^S$ is the same. Then, summing over (11) and (12) for any $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (-\infty, \tau_{k+1}]$:

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \infty)} \left[ \max \left\{ v\left(\theta, \mathbf{q}^{\tilde{\beta}}(\theta), \mathbf{p}^{\tilde{\beta}}(\theta)\right), 0 \right\} - \max \left\{ v\left(\theta, \mathbf{q}^{\tilde{\beta}}(\theta'), \mathbf{p}^{\tilde{\beta}}(\theta')\right), 0 \right\} \right] \mu_0(\theta) \geq 0 \tag{13}$$

which is the condition that guarantees there are no downward uniform deviations at $\tilde{h}^B_k$. Similarly, from (11), for any $\theta' \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_{k+1}, \tau_k]$:

$$\sum_{\theta \in \hat{\Theta}(h^B_{k-1}) \cap (\tau_k, \infty)} \left[ \max \left\{ v\left(\theta, \mathbf{q}^{\tilde{\beta}}(\theta), \mathbf{p}^{\tilde{\beta}}(\theta)\right), 0 \right\} - \max \left\{ v\left(\theta, \mathbf{q}^{\tilde{\beta}}(\theta'), \mathbf{p}^{\tilde{\beta}}(\theta')\right), 0 \right\} \right] \mu_0(\theta) \geq 0 \tag{14}$$

which is the condition that guarantees there are no downward uniform deviations at $\tilde{h}^B_{k+1}$.

**Step 3**. Now I alter all succeeding histories.

**Step 3a**. Now consider any $h^S_l = (\boldsymbol{\tau}_l, \mathbf{m}_{l-1})$ which succeeds $h^S_k$ in the original game and is on path of a buyer with a positive signal about $\tau_k$: that is, $m_k = \beta((h^S_k; \mathbf{s}_{k-1}, a))$. If $h^S_l$ is a successor of $h^S_{k+1}$, then in a new extensive $\tilde{H}^S$ form, I change the order of questions and include

$$\tilde{h}^S_l = (\boldsymbol{\tau}_{k-1}, \underbrace{\tau_{k+1}, \tau_k}_{\text{Change order of thresholds}}, \boldsymbol{\tau}_{[k+2,l]}; \mathbf{m}_{k-1}, m^a, m^a, \mathbf{m}_{[k,l-1]})$$

Otherwise, I insert an extra threshold into $h^S_l$ and include:

$$\tilde{h}^S_{l+1} = (\boldsymbol{\tau}_{k-1}, \underbrace{\tau_{k+1}, \tau_k}_{\text{Include extra threshold } \tau_{k+1} \text{ before } \tau_k}, \boldsymbol{\tau}_{[k+2,l]}; \mathbf{m}_{k-1}, m^a, m^a, \mathbf{m}_{[k-1,l-1]})$$

Given Step 2, a buyer plausible history $\tilde{h}^B_{l+1} = (\tilde{h}^S_{l+1}, \mathbf{s}_{l+1})$ is only on-path if the buyer gets a positive signal about both thresholds $\tau_k$ and $\tau_{k+1}$: $s_k = s_{k+1} = a$.[15] For every such $\tilde{h}^B_{l+1}$, we can consider a buyer plausible history with the same signal realizations for coinciding thresholds $h^B_l = (h^S_l; \mathbf{s}_k, \mathbf{s}_{[k+1,l]})$. Note that $\hat{\Theta}(\tilde{h}^B_{l+1}) = \hat{\Theta}(h^B_l)$ — the buyer has the same information at these two histories because a positive signal about $\tau_k$ only is as good as two positive signals about $\tau_k$ and $\tau_{k+1}$. Define the new strategy so that the action at these two histories coincides, as well: $\tilde{\beta}(\tilde{h}^B_{l+1}) = \beta(h^B_l)$. Then, if Assumption 1 is satisfied, then the restrictions of (D-IC) in the new mechanism for history $\tilde{h}^B_{l+1}$ coincides with those of the old mechanism at $h^B_l$.

**Step 3b**. For every $h^S_l = (\boldsymbol{\tau}_l, \mathbf{m}_l)$ which succeeds $h^S_{k+1}$ and is on the path of a buyer who gets a positive signal with threshold $\tau_{k+1}$ in the original game, I essentially only change the order of thresholds. That is, for $h^S_l$ which has $m_{k+1} = \beta((h^S_{k+1}; \mathbf{s}_k, a))$, I let $\tilde{H}^S$ include

$$\tilde{h}^S_l = (\boldsymbol{\tau}_{k-1}, \underbrace{\tau_{k+1}, \tau_k}_{\text{Change order of thresholds}}, \boldsymbol{\tau}_{[k+2,l]}; \mathbf{m}_{k-1}, m^a, m^b, \mathbf{m}_{[k-1,l-1]})$$

Analogous to Step 3a, $\tilde{h}^B_l = (\tilde{h}^S_l, \mathbf{s}_l)$ is only on-path of the buyer types who get a positive signal

[15]The case when the replacement is $\tilde{h}^S_l$ is analogous.

about $\tau_{k+1}$, but negative about $\tau_k$. The new buyer's strategy is defined so that the buyer reports the same message at $\tilde{h}^B_l$ as in the original mechanism, at history $h^B_l = (h^S_l; \mathbf{s}_{k-1}, s_{k+1}, s_k, \mathbf{s}_{[k+2,l]})$: $\tilde{\beta}(\tilde{h}^B_l) = \beta(h^B_l)$. Again, information at these two histories is the same, implying that under Assumption 1, there is no profitable myopic downward deviation at $\tilde{h}^B_l$ as long as there is no such deviation at $h^B_l$.

**Step 3c**. Similarly, for every $h^S_l = (\boldsymbol{\tau}_l, \mathbf{m}_l)$ succeding $h^S_{k+1}$ which is on the path of a buyer with a negative signal about threshold $\tau_{k+1}$, I delete the question about $\tau_k$ (as it is useless). Formally, if $h^B_l = (\boldsymbol{\tau}_l, \mathbf{m}_l)$ succeeds $h^B_{k+1}$ in the original game with $m_{k+1} = \beta((h^S_{k+1}; \mathbf{s}_k, b))$, then $\tilde{H}^S$ includes a history

$$\tilde{h}^S_{l-1} = (\underbrace{\boldsymbol{\tau}_{k-1}, \tau_{k+1}}_{\text{Skip the question about } \tau_k}, \boldsymbol{\tau}_{[k+2,l]}; \mathbf{m}_{k-1}, m^b, \mathbf{m}_{[k+2,l-1]})$$

Again similar to Step 3a, a history $\tilde{h}^B_{l-1} = (\tilde{h}^S_{l-1}, \mathbf{s}_{l-1})$ is only on-path for a buyer who observes a negative signal about $\tau_{k+1}$. The new buyer's strategy is defined so that the buyer reports the same message at $\tilde{h}^B_{l-1}$ as in the original mechanism, at history $h^B_l = (h^S_l; \mathbf{s}_k, b, \mathbf{s}_{[k+1,l-1]})$: $\tilde{\beta}(\tilde{h}^B_{l-1}) = \beta(h^B_l)$. By the same argument as before, there is no profitable myopic downward deviation at $\tilde{h}^B_{l-1}$.

The construction of new selling stage histories and buyer's strategies at these histories is analogous. All the histories that are not considered in Steps 1-3 are off-path, and $\tilde{\beta}$ at these can be defined arbitrarily with no impact on the seller's payoff or the constraints. Now I have completed the description of the new mechanism $\langle \tilde{H}^S, \tilde{\beta} \rangle$. By construction, it satisfies Assumption 1, so that the conclusions about myopic downward deviations are valid: the new mechanism is feasible. Moreover, if the outcome allocations are the same, the seller gets the same profit from $\langle \tilde{H}^S, \tilde{\beta} \rangle$ as in the original mechanism.

□

**Lemma 5** (Up or Out Mechanisms)**.** *It is without loss of optimality to consider*

*(i) extensive forms $H^S$, which communicate about thresholds in increasing order, and for every communication stage history $h^S_k = (\boldsymbol{\tau}_k; \mathbf{m}_{k-1}) \in H^S_C$, the buyer can proceed to a selling stage: exists $h^S_{k,o} = (\boldsymbol{\tau}_k; \mathbf{m}_{k-1}, m^b; q, p) \in H^S_O$ with some offer $(q, p)$*

*(ii) buyer strategies $\beta$ is truthful: $\beta((\boldsymbol{\tau}_k; \mathbf{m}_{k-1}; \mathbf{s}_k)) = m^{s_k}$.*

*Proof.* First, impose Lemma 4 and only consider extensive forms which communicate about thresholds in increasing order. I will now show that for any $\langle H^S, \beta \rangle$, it is possible to find a feasible mechanism $\langle \tilde{H}^S, \tilde{\beta} \rangle$ that satisfies properties of the lemma. I prove this statement by induction.

**Step 1**. First, I show that it is without loss to consider mechanisms, where the buyer leaves after the negative signal about the first threshold. Without loss, suppose the messages are labeled so that a buyer with a positive signal reports $m^a$: $\beta(\tau_k, a) = m^a$. Since a buyer type $\theta$ with a negative signal about $\tau_1$ ($\theta < \tau_1$) expects to get no further useful information, he is indifferent between following $\beta$ in $H^S$ or purchasing $(\mathbf{q}^{H^S_;\beta}(\theta), \mathbf{p}^{H^S_;\beta}(\theta))$ right away. In other words, I consider $\tilde{H}^S$ which contains $\tilde{h}^S_{1,o} = (\tau_1, m^b; \mathbf{q}^{H^S_;\beta}(\theta), \mathbf{p}^{H^S_;\beta}(\theta))$. Do no other modifications, meaning if a seller's history $h^S \in H^S$ has $m_1 = m^a$, then it is included in the new extensive form.

Under the new strategy $\tilde{\beta}$, the buyer reports $m^b$ after a negative signal about $\tau_1$: $\tilde{\beta}(\tau_1, b) = m^b$ and makes the same purchasing decision about the offer, as at respective outcome of an original mechanism: $\tilde{\beta}(\tau_1; m^b; \mathbf{q}^{H^S_;\beta}(\theta), \mathbf{p}^{H^S_;\beta}(\theta); b, \sigma(\mathbf{q}^{H^S_;\beta}(\theta)/\mathbf{p}^{H^S_;\beta}(\theta))) = \beta(\omega^\beta(\theta))$. And for all buyer plausible histories which coincide in both extensive forms, let the buyer choose the same action: $\tilde{\beta}(h^B) = \beta(h^B), \forall h^B \in H^B \cap \tilde{H}^B$.

By construction, the modified mechanism has the same outcome allocation as the original one and is also (IR). To check that there are no profitable myopic downward deviations , note that both $\tilde{H}^S$ and $H^S$ have exactly the same on-path histories with a positive last signal realization and the same outcome allocation. So if $\langle H^S, \beta \rangle$ is (D-IC), so is the new mechanism.

**Step** $l+1$. Suppose that for all communication stage histories of length not exceeding $l$, the properties of the lemma are true. Since all the thresholds are communicated in increasing order, the set of types remaining in communication in a history of length $l+1$ is $\hat{\Theta} \cap (\tau_l, \infty)$, who all observe the same history induced at $h^S_l$: $h^B_l = (h^S_l; a, \dots, a)$. At $h^S_{l+1}$, on path the buyer either observes $h^B_{l+1,a} = (h^S_{l+1}; a, \dots, a, a)$ or $h^B_{l+1,b} = (h^S_{l+1}; a, \dots, a, b)$. Again, I can, if necessary, relabel the messages so that $\beta(h^B_{l+1,a}) = m^a$. Let $\tilde{H}^S$ contain any history with $m_l = m^a$ so that the buyer types with a positive signal about $\tau_{l+1}$ have the exact same continuation game. Analogous to Step 1, I also include a history $(\boldsymbol{\tau}_{l+1}; \mathbf{m}_l, m^b; \mathbf{q}^{H^S_;\beta}(\theta), \mathbf{p}^{H^S_;\beta}(\theta))$ in the new extensive form $\tilde{H}^S$, where $\theta \in \Theta \cap (\tau_l, \tau_{l+1}]$ and preserve the same purchasing decision about the offer as in the original extensive form. By the same reasoning as in Step 1, $\langle \tilde{H}^S, \tilde{\beta} \rangle$ must be feasible if the original mechanism is feasible and it delivers the same profit. Moreover, it satisfies the properties of the lemma for all communication stage histories of the length not exceeding $l + 1$. This completes the argument of induction.

□

**Proof of Lemma 1 :** Consider some mechanism $\langle H^S, \beta \rangle$ with the properties of Lemma 5. I will now show that it is possible to construct a new mechanism, $\langle \tilde{H}^S, \tilde{\beta} \rangle$, which has the same outcome allocation and features bottom-up communication and truthful reports of a buyer.

**Step 1**. First, suppose that there are some communication stage histories $h_k^S = (\boldsymbol{\tau}_k; m^a, \ldots, m^a)$ and $h_k^S = (\boldsymbol{\tau}_{k+1}; m^a, \ldots, m^a)$, such that there is no $\theta \in \Theta \cap (\tau_k, \tau_{k+1}]$. By the properties of the mechanism, the buyer remains in communication by $h_{k+1}^S$ only when he observes positive signals about all prior thresholds. And if the set $\Theta \cap (\tau_k, \tau_{k+1}]$ is empty, then all the buyers who stay in communication by $h_{k+1}^B$ observe the same signal realization $a$ and report the same message $m^a$.

Then, it is possible to cut out the threshold $\tau_{k+1}$ from all succeeding histories without influencing any choice-relevant valuables. I describe the new mechanism formally below.

Suppose a seller's history $h^S$ is not $h_{k+1}^S$ and does not succeed it, then $h^S$ is included in $\tilde{H}^S$. If a communication stage history $h_l^S = (\boldsymbol{\tau}_l; m^a, \ldots, m^a)$ succeeds $h_{k+1}^S$, then $\tilde{H}^S$ includes $\tilde{h}_{l-1}^S = (\boldsymbol{\tau}_k, \boldsymbol{\tau}_{[k+2,l]}; m^a, \ldots, m^a)$. Similarly, if a selling stage history $h_{l,o}^S = (\boldsymbol{\tau}_l; m^a, \ldots, m^a, m_l; q, p)$ succeeds $h_{k+1}^S$, then $\tilde{H}^S$ contains $\tilde{h}_{l-1,o}^S = (\boldsymbol{\tau}_k, \boldsymbol{\tau}_{[k+2,l]}; m^a, \ldots, m^a, m_l; q, p)$. By construction, a truthful strategy in $\tilde{H}^S$ leads to the same outcome allocation as in the original mechanism. To check (D-IC), consider deviations at some on-path plausible buyer history $\tilde{h}_l^B$. Note that $\tilde{h}_l^B$ is of the form $(\tilde{\boldsymbol{\tau}}_l; a, \ldots, a)$. Either $\tilde{h}_l^B$ itself is on-path in the original mechanism, or $h_{l+1}^B = (\tilde{\boldsymbol{\tau}}_k, \tau_{k+1}, \tilde{\boldsymbol{\tau}}_{[k+1,l]}; a, \ldots, a)$ is and contains the information: $\hat{\Theta}(h_{l+1}^B) = \hat{\Theta}(\tilde{h}_l^B)$ (by assumption). Since there is no profitable myopic downward deviation at $h_{l+1}^B$, and allocations are the same, there is no such deviation at $\tilde{h}_l^B$, too.

**Step 2**. Suppose that there are two types $\{\theta_i, \theta_{i+1}\}$, which get pooled by the seller. Formally, suppose that there is a communication stage history $h_k^S = (\boldsymbol{\tau}_k; m^a, \ldots, m^a) \in H_k^S$, such that $\tau_{k-1} < \theta_i < \theta_{i+1} \leq \tau_k$. We can modify the mechanism so that the seller first uses a threshold equal to $\theta_i$ before asking about $\tau_k$, with both types $\theta_i$ and $\theta_i + 1$ getting the same induced allocation.

Formally, let $\tilde{H}^S$ contain every $h^S$ which is not $h_k^S$ and does not succeed it. Every communication stage history $h_l^S = (\boldsymbol{\tau}_l; m^a, \ldots, m^a) \supset h_k^S$ from the original extensive form gets replaced by $\tilde{h}_{l+1}^S = (\boldsymbol{\tau}_k, \theta_i, \boldsymbol{\tau}_{[k+1,l]}; m^a, \ldots, m^a)$. Similarly, every selling stage history $h_{l,o}^S = (\boldsymbol{\tau}_l; m^a, \ldots, m^a, m_l; q, p) \supset h_k^S$ gets replaced by $\tilde{h}_{l+1,o}^S = (\boldsymbol{\tau}_k, \theta_i, \boldsymbol{\tau}_{[k+1,l]}; m^a, \ldots, m^a, m_l)$. Additionally, let $\tilde{H}_O^S$ contain $(\boldsymbol{\tau}_{k-1}, \theta_i; m^a, \ldots, m^a)$ and $\tilde{h}_k^S = (\boldsymbol{\tau}_{k-1}, \theta_i; m^a, \ldots, m^a, m^b; \mathbf{q}^{H_;^S\beta}(\theta_i), \mathbf{p}^{H_;^S\beta}(\theta_i))$. Just as in the previous step, if the buyer reports truthfully, the outcome allocation in the new extensive form is the same; and there is no profitable myopic downward deviation at communication histories of length not equal to $k+1$. It only remains to verify that there is no such deviation at a history where the buyer learns his type is above $\theta_i$ but below $\tau_k$: $\tilde{h}_k^B = (\boldsymbol{\tau}_{k-1}, \theta_i, \tau_k; m^a, \ldots, m^a; a, \ldots, a)$. At this history, the only available downward deviation is towards $\theta_i$. Since there is no profitable deviation at $h_k^B = (h_k^S; a \ldots, a)$:

$$\sum_{\theta > \tau_k} \max\left\{v\left(\theta, \mathbf{q}^{H_;^S\beta}(\theta), \mathbf{p}^{H_;^S\beta}(\theta)\right), 0\right\}\mu_0(\theta) \geq \sum_{\theta > \tau_k} \max\left\{v\left(\theta, \mathbf{q}^{H_;^S\beta}(\theta_i), \mathbf{p}^{H_;^S\beta}(\theta_i)\right), 0\right\}\mu_0(\theta)$$

Since two types $\theta_i, \theta_{i+1}$ are pooled, they get the same outcome allocation under the original mechanism, so that we obtain:

$$\sum_{\theta > \tau_k} \max\left\{v\left(\theta, \mathbf{q}^{H_;^S\beta}(\theta), \mathbf{p}^{H_;^S\beta}(\theta)\right), 0\right\}\mu_0(\theta) + \max\left\{v\left(\theta_{i+1}, \mathbf{q}^{H_;^S\beta}(\theta_{i+1}), \mathbf{p}^{H_;^S\beta}(\theta_{i+1})\right)\right\}\mu_0(\theta_{i+1}) \geq$$
$$\sum_{\theta > \tau_k} \max\left\{v\left(\theta, \mathbf{q}^{H_;^S\beta}(\theta), \mathbf{p}^{H_;^S\beta}(\theta)\right), 0\right\}\mu_0(\theta) + \max\left\{v\left(\theta_{i+1}, \mathbf{q}^{H_;^S\beta}(\theta_i), \mathbf{p}^{H_;^S\beta}(\theta_i)\right), 0\right\}\mu_0(\theta_{i+1})$$
$$\sum_{\theta \in \hat{\Theta}(\tilde{h}_k^B)} \max\left\{v\left(\theta, \mathbf{q}^{H_;^S\beta}(\theta), \mathbf{p}^{H_;^S\beta}(\theta)\right), 0\right\}\mu_0(\theta) \geq \sum_{\theta \in \hat{\Theta}(\tilde{h}_k^B)} \max\left\{v\left(\theta, \mathbf{q}^{H_;^S\beta}(\theta_i), \mathbf{p}^{H_;^S\beta}(\theta_i)\right), 0\right\}\mu_0(\theta)$$

□

# B Proofs for Section 5.3.2

**Proof of Lemma 2 :** If $\theta < \theta_n \leq c$, then the seller does not serve the whole market and the statement of the lemma is true. Now suppose $\theta_n > c$. As there is no D-IC$_i$ at $i = n$, it must

$\mathbf{q}^*(\theta_n) = 1$. The statement can then only be violated if $\exists i \in \{1, \ldots, N-2\}$ such that type $\theta_i$ is served a premium quality, but the quality of a higher type is rationed $\mathbf{q}^*(\theta_{i+1}) < 1$. Suppose this is the case, then, by Lemma 9, $\text{IR}_{i+1}$ binds implying that $\sum_{j>i}^{N} U^*(\theta_j)\mu_0(\theta_j) = \sum_{j>i+1}^{N} U^*(\theta_j)\mu_0(\theta_j)$. If $U^*, \mathbf{q}^*$ is feasible, then from D-IC$_i$:

$$\sum_{j>i}^{n} U^*(\theta_j)\mu_0(\theta_j) \geq \sum_{j>i}^{n} \left[U^*(\theta_i) + (\theta_j - \theta_i)\right] \mu_0(\theta_j)$$

$$\implies \sum_{j>i+1}^{n} U^*(\theta_j)\mu_0(\theta_j) = \sum_{j>i}^{n} U^*(\theta_j)\mu_0(\theta_j) \geq \sum_{j>i}^{n} (\theta_j - \theta_i)\mu_0(\theta_j) > \sum_{j>i+1}^{n} (\theta_j - \theta_{i+1})\mu_0(\theta_j)$$

$$\implies \sum_{j>i+1}^{n} U^*(\theta_j)\mu_0(\theta_j) > \sum_{j>i+1}^{n} \left[U^*(\theta_{i+1}) + \mathbf{q}^*(\theta_{i+1})(\theta_j - \theta_{i+1})\right] \mu_0(\theta_j)$$

Then, D-IC$_{i+1}$ is slack which contradicts Lemma 9. □

# C Proofs for Section 7

In this section, I present omitted proofs for the generalized preferences model described in Section 7.

**Lemma 6.** *Both* $\mathbf{q}^e(\cdot)$ *and* $\mathbf{q}^d(\cdot, CS)$ *are increasing functions.*

*Proof.* Using monotone comparative statics theorem from Milgrom & Shannon (1994), as $u(\cdot, \cdot)$ satisfies increasing differences, $\mathbf{q}^e(\cdot)$ is increasing. Now I prove of $\mathbf{q}^d(\cdot, CS)$ is also increasing.

Suppose by contradiction that there exist $j > i$, such that $\mathbf{q}^d(\theta_j, CS) < \mathbf{q}^d(\theta_i, CS) \leq \infty$. By increasing differences of $u(\cdot, \cdot)$:

$$u(\theta, \mathbf{q}^d(\theta_i, CS)) - u(\theta_i, \mathbf{q}^d(\theta_i, CS)) > u(\theta, \mathbf{q}^d(\theta_j, CS)) - u(\theta_i, \mathbf{q}^d(\theta_j, CS)), \forall \theta > \theta_i \quad (15)$$

Note that assumptions (i) and (iii) about $u(\cdot, \cdot)$ imply $u(\cdot, q)$ is increasing for every $q$. Combining this observation with Inequality (15), we obtain:

$$CS \geq \mathbb{E}\left[\left(u(\theta, \mathbf{q}^d(\theta_i, CS)) - u(\theta_i, \mathbf{q}^d(\theta_i, CS))\right)_+\right] > \mathbb{E}\left[\left(u(\theta, \mathbf{q}^d(\theta_j, CS)) - u(\theta_i, \mathbf{q}^d(\theta_j, CS))\right)_+\right]$$
$$> \mathbb{E}\left[\left(u(\theta, \mathbf{q}^d(\theta_j, CS)) - u(\theta_j, \mathbf{q}^d(\theta_j, CS))\right)_+\right]$$

which contradicts to the definition of $\mathbf{q}^d(\cdot)$. $\square$

**Lemma 7.** *Suppose that $\{(q_i, p_i)\}_{i=1}^n$ features surplus-based distortion. If the offer $(q_i, p_i)$ does not extract the full surplus of type $\theta_i$, then quality $i$ is efficient.*

*Proof.* To prove the lemma, we need to show that for every $i \geq K(CS)$ : $q^d(\theta_i, CS) \geq q^e(\theta_i)$. As $CS^{MR}(\cdot)$ is decreasing, $CS \geq CS^{MR}(\theta_i)$ for every $i \geq K(CS)$. As $u(\cdot, \cdot)$ satisfies increasing differences and $\mathbf{q}^e(\cdot)$ is an increasing function, we obtain:

$$CS \geq CS^{MR}(\theta_i) \geq \sum_{j=i}^{n-1} \Pr(\theta > \theta_j) \left[u(\theta_{j+1}, \mathbf{q}^e(\theta_i)) - u(\theta_j, \mathbf{q}^e(\theta_i))\right] =$$

$$\sum_{j=i+1}^{n-1} \mu_0(\theta_j) u(\theta_j, \mathbf{q}^e(\theta_i)) - u(\theta_i, \mathbf{q}^e(\theta_i))) = \mathbb{E}\left[(u(\theta, \mathbf{q}^e(\theta_i)) - u(\theta_i, \mathbf{q}^e(\theta_i)))_+\right] \tag{16}$$

As $u(\cdot, \cdot)$ satisfies increasing differences, the function $\mathbb{E}\left[(u(\theta, q)) - u(\theta_i, q)))_+\right]$ is increasing in $q$. Using the definition of $\mathbf{q}^d(\theta_i, CS)$ and Inequality (16), we get $\mathbf{q}^d(\theta_i, CS) \geq \mathbf{q}^e(\theta_i)$ as desired. $\square$

**Lemma 8.** *Suppose an extensive form $H^S$ features bottom-up communication and surplus-based distortion. Then, it is optimal for the buyer to leave communication after observing a signal realization "below" for the first time and to stay in communication whenever no such signal realization is observed.*

*Proof.* First, note that since for any $i \geq K(CS)$ the pricing after round $i$ is determined by the envelope condition of Mussa & Rosen (1978), $\theta_l$ prefers $(q_l, p_l)$ among all $\{(q_l, p_l)\}_{l=K(CS)}^n$. Hence, upon reaching round $K(CS)$, each buyer type prefers to terminate communication after getting the first "below" signal realization. It remains to verify the buyer is willing to follow the strategy in all previous rounds.

Consider any round $i < K(CS)$. By definition, a selling stage offer after this round extracts the full surplus of type $\theta_i$. Since $H^S$ communicates from the bottom-up, in round $i$ the seller discloses information about threshold $\theta_i$. Conditional on getting a signal realization "below" for the first time, the buyer learns his type is $\theta_i$. If the buyer leaves immediately, he gets his surplus extracted and receives zero payoff. Otherwise, the buyer can reach some selling stage offer after some round

$j > i$. For any round $j < K(CS)$, $p_j = u(\theta_j, q_j)$

$$p_j - u(\theta_i, q_j) = u(\theta_j, q_j) - u(\theta_i, q_j) > u(\theta_j, 0) - u(\theta_i, 0) = 0$$

where the inequality is due to increasing differences assumption. Hence, the deviation towards any such selling stage offer is not beneficial. Now consider deviations towards the offer after $K(CS)$. By Lemma 7, the offered quality in the respective selling stage is efficient for $\theta_{K(CS)}$: $q_{K(CS)} = q^e\left(\theta_{K(CS)}\right)$. By definition of $K(CS)$, $p_{K(CS)}$ is at least

$$u\left(\theta_{K(CS)-1}, q^e\left(\theta_{K(CS)-1}\right)\right) + u\left(\theta_{K(CS)}, q^e\left(\theta_{K(CS)}\right)\right) - u\left(\theta_{K(CS)}, q^e\left(\theta_{K(CS)-1}\right)\right)$$

Then, a type $\theta_i$'s payoff of deviating towards offer at round $K(CS)$ cannot be beneficial, as:

$$\begin{aligned} u(\theta_i, q_{K(CS)}) - p_{K(CS)} < u(\theta_{K(CS)-1}, q_{K(CS)}) - p_{K(CS)} = \\ u\left(\theta_{K(CS)-1}, \mathbf{q}^e\left(\theta_{K(CS)}\right)\right) - u\left(\theta_{K(CS)}, \mathbf{q}^e\left(\theta_{K(CS)}\right)\right) - \\ \left[u\left(\theta_{K(CS)-1}, \mathbf{q}^e\left(\theta_{K(CS)-1}\right)\right) - u\left(\theta_{K(CS)}, \mathbf{q}^e\left(\theta_{K(CS)-1}\right)\right)\right] < 0 \end{aligned}$$

since $\mathbf{q}^e(\cdot)$ is an increasing function and $u(\cdot, \cdot)$ satisfies increasing differences. In particular,

$$u(\theta_{K(CS)}, q_{K(CS)}) - p_{K(CS)} \geq u(\theta_{K(CS)}, q_j) - p_j$$

By increasing differences and envelope pricing in all following rounds, any deviation towards any later offers is also unprofitable.

Now suppose the buyer gets a positive signal about $\theta_i$ in a round $i < K(CS)$. Conditional on learning the buyer type is above $\theta_i$, the expected surplus of staying in communication is $CS/\Pr(\theta > \theta_i)$. Alternatively, the buyer's (conditional) expected surplus from leaving communication immediately after round $i$ is:

$$\mathbb{E}[u(\theta, q_i) - p_i | \theta > \theta_i] = \mathbb{E}[u(\theta, q_i) - u(\theta_i, q_i) | \theta > \theta_i] = \frac{\mathbb{E}\left[\left(u(\theta, q_i) - u(\theta_i, q_i)\right)_+\right]}{\Pr(\theta > \theta_i)}$$

As $q_i = \min\{\mathbf{q}^d(\theta_i, CS), \mathbf{q}^e(\theta_i)\}$ and $u(\cdot,\cdot)$ satisfies increasing differences, $\mathbb{E}[u(\theta, q_i) - p_i | \theta > \theta_i]$ is at most $CS / \Pr(\theta > \theta_i)$. Hence, after getting a positive signal about $\theta_i$, the buyer is willing to stay in communication. This completes the proof. □

**Proof of Theorem 2 :** Lemma 1 readily applies to the setting with general preferences: bottom-up communication is optimal in a relaxed problem, which only requires the buyer's strategy to be D-IC and IR. Analogously to the linear case, I formulate the equivalent static problem to solve for an optimal allocation:

**Equivalent Static Problem (General Preferences):**

$$\max_{\mathbf{q}:\Theta\to[0,\bar{Q}],U:\Theta\to\mathbb{R}} \quad \sum_{i=1}^{n} \mu_0(\theta_i)\left[u(\theta_i, \mathbf{q}(\theta_i)) - c(\mathbf{q}(\theta_i)) - U(\theta_i)\right]$$

subject to

$$\sum_{j>i}^{n} U(\theta_j)\mu_0(\theta_j) \geq \sum_{j>i}^{n} \left[U(\theta_i) + u(\theta_j, \mathbf{q}(\theta_i)) - u(\theta_i, \mathbf{q}(\theta_i))\right]\mu_0(\theta_j) \tag{D-IC$_i$}$$

$$U(\theta_i) \geq 0 \tag{IR$_i$}$$

**Lemma 9.** *Suppose that $\mathbf{q}^*(\cdot), U^*(\cdot)$ is a solution to the Equivalent Static Problem for general preferences, then*

*(i) If $\mathbf{q}(\theta_i) \neq \mathbf{q}^e(\theta_i)$, then D-IC$_i$ is binding.*

*(ii) If $\mathbf{q}(\theta_j) < \mathbf{q}^e(\theta_j)$ for some $\theta_j \geq \theta_i$ then IR$_i$ is binding.*

*Proof.* The first part is straightforward. If $\mathbf{q}^*(\theta_i) \neq \mathbf{q}^e(\theta_i)$ and D-IC$_i$ slack, it is feasible for the seller marginally $\mathbf{q}(\theta_i)$ towards $\mathbf{q}^e(\theta_i)$. This change improves upon seller's profit, since $u(\theta, \cdot) - c(\cdot)$ is concave for every $\theta$.

Now I prove the second part. Suppose by a way of contradiction, there exist $\theta_k$ and $\theta_l \geq \theta_k$, such that $\mathbf{q}^*(\theta_l) < \mathbf{q}^e(\theta_l)$ and $U^*(\theta_k) > 0$. By part (i), $\mathbf{q}(\theta_n) = \mathbf{q}^e(\theta_n)$, as there is no downward incentive constraint for the highest type. Hence, if the premise is true, then $j < n$ and it is possible to define $\tilde{U}, \tilde{q}$, so that:

$$\tilde{U}(\theta_i) = \begin{cases} U^*(\theta_i) + \varepsilon \dfrac{\mu_0(\theta_k)}{\Pr(\theta > \theta_k)}, \text{ if } i > j \\ U^*(\theta_i) - \varepsilon, \text{ if } i = k \\ U^*(\theta_i), \text{ else} \end{cases} \qquad \tilde{\mathbf{q}}(\theta_i) = \begin{cases} \mathbf{q}^*(\theta_i), \text{ if } i \neq j \\ \mathbf{q}^d\left(\theta_i, \sum_{j>i}^n \left(\tilde{U}(\theta_j) - \tilde{U}(\theta_i)\right)\mu_0(\theta_j)\right), \text{ if } i = l \end{cases}$$

First, note that for small enough $\varepsilon$, $\tilde{\mathbf{q}}(\cdot)$ generates a higher total surplus than $\mathbf{q}^*$ does. To prove this, it is sufficient to verify $\mathbf{q}^*(\theta_l) > \tilde{\mathbf{q}}(\theta_l) > \mathbf{q}^*(\theta_l)$. By assumption, we have $\mathbf{q}^*(\theta_l) < \mathbf{q}^e(\theta_l)$, and part (i) implies that D-IC$_l$ binds in the suggested solution, so that $\mathbf{q}^*(\theta_j) = \mathbf{q}^d(\theta_j, \sum_{j>i}^n (U^*(\theta_j) - U^*(\theta_l))\mu_0(\theta_j)) < \infty$. Note that

$$\sum_{j>i}^n (\tilde{U}(\theta_j) - \tilde{U}(\theta_l))\mu_0(\theta_j) = \sum_{j>i}^n (U^*(\theta_j) - U^*(\theta_l))\mu_0(\theta_j)) + \varepsilon\mu_0(\theta_k)$$

implies $\tilde{\mathbf{q}}(\theta_l) > \mathbf{q}^*(\theta_l)$, as $\mathbf{q}^d(\theta, \cdot)$ is strictly increasing whenever finite. Moreover, for small enough $\varepsilon$, $\tilde{\mathbf{q}}(\theta_l) < \mathbf{q}^e(\theta_l)$, as required.

Now let's verify that the suggested deviation is feasible. First, for small enough $\varepsilon$, $\tilde{U}(\theta_i) \geq 0$, meaning the deviation does not violate IR$_i$ for any $i$. To verify D-IC$_i$ for every $i \neq l$, the left-hand side of D-IC$_i$ is higher, while its right-hand side is smaller under the suggested deviation, as compared to the solution $\mathbf{q}^*(\cdot), U^*(\cdot)$. Hence, as long as the solution itself satisfies D-IC$_i$ for every $i \neq l$, so does the suggested deviation. Moreover, it also does not violate D-IC$_l$ by the definition of $\tilde{\mathbf{q}}(\theta_l)$.

The suggested deviation maintains the same consumer surplus while enhancing the total generated surplus. Consequently, it leads to an improvement in the seller's profit, which contradicts the optimality of $\mathbf{q}^*(\cdot), U^*(\cdot)$. This concludes the proof. □

From Lemma 9, if $\mathbf{q}^*(\cdot), U^*(\cdot)$ is a solution, then either $\mathbf{q}^*(\theta_i) = \mathbf{q}^e(\theta_i)$ , or D-IC$_i$ binds together with IR$_i$. Moreover, Lemma 9 implies that whenever D-IC$_i$ binds, its left-hand side must be equal to the total consumer surplus $CS = \sum_{j=1}^n U^*(\theta_j)\mu_0(\theta_j)$. Consequently, $\mathbf{q}^*(\theta_i) = \mathbf{q}^d(\theta_i, CS)$ for all such $i$. Furthemore, note that the LHS of D-IC$_i$ is at most $CS$ for any $i$, hence if this constraint is slack at $\mathbf{q}^*(\theta_i)$, it must be that $\mathbf{q}^*(\theta_i) \leq \mathbf{q}^d(\theta_i, CS)$. Then, we have $\mathbf{q}^*(\theta_i) = \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}$ for every $\theta_i$.

Finally, note that if $\mathbf{q}^*(\cdot), U^*(\cdot)$ solves the seller's problem, so does $\mathbf{q}^*(\cdot), \tilde{U}(\cdot)$ as long as it is feasible and generates the same expected consumer surplus. In particular, one can define $\tilde{U}(\cdot)$ as suggested by the outcome of surplus-based distortion:

$$\tilde{U}(\theta_i) = \begin{cases} 0, \text{ if } i < K(CS) \\ \frac{CS - CS^{MR}(\theta_i)}{\Pr(\theta \geq \theta_i)}, \text{ if } \theta_i = K(CS) \\ \frac{CS - CS^{MR}(\theta_i)}{\Pr(\theta \geq \theta_i)} + \sum_{j=K(CS)+1}^{i} u(\theta_j, \mathbf{q}^e(\theta_{j-1})) - u(\theta_{j-1}, \mathbf{q}^e(\theta_{j-1})), \text{ if } i > K(CS) \end{cases}$$

To find the optimal $CS$, one should solve the following problem:

$$\max_{CS \geq 0} \sum_{i=1}^{n} \mu_0(\theta_i) \left[ u(\theta_i, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}) - c\left(\min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}\right)\right] - CS$$

I now consider the derivative of the function above with respect to $CS$. Let $CS(\theta_i)$ denote a consumer surplus, such that $\mathbf{q}^e(\theta_i) = \mathbf{q}^d(\theta_i, CS(\theta_i))$. At $CS(\theta_i)$, the function $\min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}$ is not differentiable. If $\mathbf{q}^e(\theta_i)$ is interior, then $\lim_{CS \to CS(\theta_i)_-} u'(\theta_i, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}) = 0$, and we have:

$$\lim_{CS \to CS(\theta_i)_-} u'_q(\theta_i, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}) \frac{\partial \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}}{\partial CS} =$$
$$u'_q(\theta_i, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}) \frac{\partial \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}}{\partial CS} = 0, \forall CS > CS(\theta_i)$$

where I also use

$$\lim_{CS \to CS(\theta_i)_-} \frac{\partial \mathbf{q}^d(\theta_i, CS)}{\partial CS} = \frac{1}{\mathbb{E}\left[\left(u'_q(\theta, \mathbf{q}^e(\theta_i)) - u'_q(\theta_i, \mathbf{q}^e(\theta_i))\right)_+\right]} < \infty, \forall i < n$$

due to strict increasing differences assumption on $u(\cdot, \cdot)$. Then, I can write the derivative of the objective with respect to $CS$ as:

$$\sum_{i=1}^{K(CS)-1} \frac{u'_q(\theta_i, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}) - c'(\min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\})}{\mathbb{E}\left[\left(u'_q(\theta, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}) - u'_q(\theta_i, \min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\}\right)_+\right]} - 1$$

where I use $\min\{\mathbf{q}^e(\theta_i), \mathbf{q}^d(\theta_i, CS)\} = \mathbf{q}^e(\theta_i, CS)$ for $i \geq K(CS)$ as verified by Lemma 7. Note that for $CS$ high enough, $K(CS) = 1$ and the derivative above is negative. Hence, either the seller chooses a boundary solution at $CS = 0$, or an interior $CS$, which sets the derivative to 0. This completes the proof.

□

# D Proofs for Section 8

**Proof of Proposition 1 :** For this setting, an analog of myopic downward deviation is as follows. The buyer chooses a higher (lower) type $\theta_i$, whose horizontal allocation is $-1$ (1), and is only ruled out by the last signal at history $h_t^B$. Starting from history $h_t^B$, the buyer reports as if observing the signal realization of $\theta_i$. By the same reasoning as used in Lemma 1, the necessary conditions for $\langle \mathbf{q}^{H_i^S\beta}(\cdot), \mathbf{l}^{H_i^S\beta}(\cdot), U^{H_i^S\beta}(\cdot)\rangle$ to be incentive compatible are:

$$\sum_{\theta<\theta_i} U(\theta)\mu_0(\theta) \geq \mathbf{q}^{H_i^S\beta}(\theta) \sum_{\theta<\theta_i} \mu_0(\theta)(\theta_i - \theta) + \sum_{\theta<\theta_i} U(\theta_0)\mu_0(\theta), \text{ if } \mathbf{l}^{H_i^S\beta}(\theta_i) = -1$$
$$\sum_{\theta>\theta_i} U(\theta)\mu_0(\theta) \geq \mathbf{q}^{H_i^S\beta}(\theta) \sum_{\theta>\theta_i} \mu_0(\theta)(\theta - \theta_i) + \sum_{\theta>\theta_i} U(\theta_0)\mu_0(\theta), \text{ if } \mathbf{l}^{H_i^S\beta}(\theta_i) = 1$$

I now consider a relaxed problem that only imposes IR constraints and the necessary IC conditions as above. An equivalent static problem with respect to the induced allocation is:

$$\max_{\substack{\mathbf{q}:\Theta\to[0,1], \mathbf{l}:\Theta\to\{-1,1\},\\ U:\Theta\to\mathbb{R}}} \sum_{i=1}^{n} \mu_0(\theta_i)\left[(\bar{v} - c - |\theta_i - \mathbf{l}(\theta_i)|)\mathbf{q}(\theta_i) - U(\theta_i)\right]$$

$$\text{subject to } \sum_{\theta<\theta_i} U(\theta)\mu_0(\theta) \geq \mathbf{q}(\theta) \sum_{\theta<\theta_i} \mu_0(\theta)(\theta_i - \theta) + \sum_{\theta<\theta_i} U(\theta_0)\mu_0(\theta), \text{ if } \mathbf{l}(\theta_i) = -1 \qquad \text{(D-IC}_i^{-1}\text{)}$$

$$\sum_{\theta>\theta_i} U(\theta)\mu_0(\theta) \geq \mathbf{q}(\theta) \sum_{\theta>\theta_i} \mu_0(\theta)(\theta-\theta_i) + \sum_{\theta>\theta_i} U(\theta_0)\mu_0(\theta), \text{ if } \mathbf{l}(\theta_i)=1 \qquad (\text{D-IC}_i^1)$$

$$U(\theta_i) \geq 0, \qquad (\text{IR}_i)$$

First, notice that since there are no constraints involving the most extreme types, it must be that these types are served efficient allocation in the optimal solution.

**Lemma 10.** *Suppose $\bar{v}$ satisfies on of the following:*

(*i*) $\bar{v} \leq 1-c$

(*ii*) $\bar{v} \geq 1+c+\max\{\mathbb{E}[(\theta-\theta_i)_+]/\Pr(\theta>\theta_i)-\theta_i, \mathbb{E}[(\theta_i-\theta)_+]/\Pr(\theta<\theta_i)+\theta_i\}, \forall\theta_i \in \Theta$

*Then, in the relaxed problem of Hotelling differentiation, there exists a unique $k$, such that for $i \leq (>)k$, the left (right) horizontal quality is served $\mathbf{l}(\theta_i)=-1$ ($=1$).*

*Proof.* If $\bar{v} \leq 1-c$, it is without loss of optimality to choose an efficient product location for every type: $\mathbf{l}(\theta_i)=-1(1)$, if $\theta_i \leq (>)0$, and $\mathbf{l}(\theta_i)=1$. Otherwise, it is not profitable to serve $\theta_i$ at all.

Now consider (*ii*) is true instead. Suppose $\langle \mathbf{q}^*(\cdot), \mathbf{l}^*(\cdot), U^*(\cdot)\rangle$ is optimal and assume there exist two consecutive types, such that the horizontal quality switches in the wrong direction: $\mathbf{l}(\theta_j)=1$ and $\mathbf{l}(\theta_{j+1})=-1$. Suppose by contradiction there exists such $j$, where $\theta_{j+1} \geq 0$. [16]

**Step 1**. First, I verify that $\mathbf{q}^*(\theta_j), \mathbf{q}^*(\theta_{j+1}) < 1$. Suppose $\mathbf{q}^*(\theta_j)=1$. Then, it is feasible for the seller to implement the efficient outcome for all types to the right of $\theta_j$ without changing the expected consumer surplus, and preserving all the incentives for the remaining types. Indeed, the following deviation $\langle \tilde{\mathbf{q}}(\cdot), \tilde{\mathbf{l}}(\cdot), \tilde{U}(\cdot)\rangle$ is feasible:

$$\tilde{\mathbf{q}}(\theta_i) = \begin{cases} \mathbf{q}^*(\theta_i), \text{ if } i \leq j \\ 1, \text{ if } i > j \end{cases} \qquad \tilde{\mathbf{l}}(\theta_i) = \begin{cases} \mathbf{l}^*(\theta_i), \text{ if } i \leq j \\ 1, \text{ if } i > j \end{cases}$$

[16] By a symmetric argument, it cannot be the case that $\mathbf{l}(\theta_j)=1$ and $\mathbf{l}(\theta_{j+1})=-1$ for some $j \leq 0$.

$$\tilde{U}(\theta_i) = \begin{cases} U^*(\theta_i), \text{ for all } i < j \\ 0, \text{ for all } n > i \geq j \\ \sum_{\theta > \theta_j} U^*(\theta)\mu_0(\theta_j)/\mu_0(\theta_i), \text{ if } i = n \end{cases}$$

The deviation is profitable since the expected consumer surplus remains the same while the total consumer surplus increases. Hence, $\mathbf{q}^*(\theta_j) < 1$ if it is part of the relaxed problem solution. The assumption about $\bar{v}$ guarantees it is efficient to serve all types, regardless of location. In this case, $\mathbf{q}^*(\theta_j)$ is interior and D-IC$^1_j$ binds.

**Step 2**. Next, I verify that $U^*(\theta_j) = 0$. Suppose by contradiction $U^*(\theta_j) > 0$, and consider a deviation towards $\tilde{U}^{\varepsilon,\theta_j}(\cdot)$:

$$\tilde{U}^{\varepsilon,\theta_j}(\theta_i) = \begin{cases} U^*(\theta_i), \text{ for all } i \notin \{1, j, n\} \\ U^*(\theta_j) - \varepsilon \text{ for all } i = j \\ U^*(\theta_i) + \varepsilon\mu_0(\theta_j)/\mu_0(\theta_i), \text{ if } i \in \{1, n\} \end{cases}$$

With this deviation, the initial allocation is feasible for all types. Moreover, the vertical quality for $\theta_j$ can be increased by $\varepsilon \cdot \Pr(\theta \geq \theta_j)/\mathbb{E}[(\theta - \theta_j)_+]$. A feasible change in profit is then:

$$\mu_0(\theta_j) \left[\varepsilon \cdot (\bar{v} - c - 1 + \theta_j) \cdot \Pr(\theta \geq \theta_j)/\mathbb{E}[(\theta - \theta_j)_+] - \varepsilon\right]$$

where I take into account that the consumer surplus is increased by $\varepsilon\mu_0(\theta_j)$ with the deviation. Given an assumption about $\bar{v}$, the change if profit is positive for every $\theta_j$, since $\mathbb{E}[(\theta - \theta_j)_+] \leq \Pr(\theta > \theta_j)(1 - \theta_j)$. As we increase $\varepsilon$, either $\tilde{\mathbf{q}}(\theta_j)$ reaches the efficient level, or $\varepsilon$ hits $U^*(\theta_j)$. The former case contradicts the first step of the proof. I conclude that is must be that $U^*(\theta_j) = 0$.

**Step 3**. As $U^*(\theta_j) = 0$, it does not affect the incentives of any other type. As type $\theta_j$ is allocated the horizontal quality on the right, it must be that:

$$\frac{\sum_{\theta > \theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta - \theta_j)_+]}(\bar{v} - c - 1 + \theta_j) \geq \min\left\{\frac{\sum_{\theta < \theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta_j - \theta)_+]}, 1\right\}(\bar{v} - c - 1 - \theta_j) \qquad (17)$$

First, suppose that $\min\left\{\frac{\sum_{\theta<\theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta_j-\theta)_+]}, 1\right\} = 1$. In this case, Equation (17) implies:

$$\frac{\sum_{\theta>\theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta-\theta_i)_+]}(\bar{v}-c-1+\theta_i) > \frac{\sum_{\theta>\theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta-\theta_j)_+]}(\bar{v}-c-1-\theta_j) >$$
$$(\bar{v}-c-1-\theta_i) \geq \mathbf{q}^*(\theta_i)(\bar{v}-c-1-\theta_i), \forall i > j$$

But then the seller can improve upon $\mathbf{q}^*, \mathbf{l}^*, U^*$ with a feasible deviation: $\langle \tilde{\mathbf{q}}(\cdot), \tilde{\mathbf{l}}(\cdot), \tilde{U}(\cdot)\rangle$ :

$$\tilde{\mathbf{q}}(\theta_i) = \begin{cases} \mathbf{q}^*(\theta_i), \text{ if } i \leq j \\ \min\left\{\frac{\sum_{\theta>\theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta-\theta_i)_+]}, 1\right\}, \text{ if } i > j \end{cases} \qquad \tilde{\mathbf{l}}(\theta_i) = \begin{cases} \mathbf{l}^*(\theta_i), \text{ if } i \leq j \\ 1, \text{ if } i > j \end{cases}$$

$$\tilde{U}(\theta_i) = \begin{cases} U^*(\theta_i), \text{ for all } i < j \\ 0, \text{ for all } n > i \geq j \\ \sum_{\theta>\theta_j} U^*(\theta)\mu_0(\theta_j)/\mu_0(\theta_i), \text{ if } i = n \end{cases}$$

Finally, consider the scenario where $\sum_{\theta<\theta_j} U^*(\theta)\mu_0(\theta)/\mathbb{E}[(\theta_j-\theta)_+] < 1$. In this case, $\mathbf{q}(\theta_{j+1})$ must be interior. By a symmetric argument used at Step 2, $U^*(\theta_{j+1}) = 0$: the seller would rather increase the vertical quality at $\mathbf{q}(\theta_{j+1})$ at the expense of increasing the expected consumer surplus. Finally, notice that as long as $U^*(\theta_j) = U^*(\theta_{j+1}) = 0$, it must be that

$$\frac{\sum_{\theta>\theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta-\theta_{j+1})_+]}(\bar{v}-c-1+\theta_{j+1}) \leq \frac{\sum_{\theta<\theta_j} U^*(\theta)\mu_0(\theta)}{\mathbb{E}[(\theta_{j+1}-\theta)_+]}(\bar{v}-c-1-\theta_{j+1})$$

or else the seller could benefit by changing the horizontal quality for $\theta_{j+1}$. We obtain a contradiction to Equation (17). This concludes the proof of the lemma. □

Given Lemma 10, the problem can be reduced to the following one:

$$\max_{k\in\{1,\ldots,n\}} \max_{\substack{\mathbf{q}:\Theta\to[0,1] \\ U:\Theta\to\mathbb{R}}} \sum_{i=1}^{n} \mu_0(\theta_i)\left[(\bar{v}-c-|\theta_i-\mathbf{l}(\theta_i)|)\mathbf{q}(\theta_i) - U(\theta_i)\right]$$

$$\text{subject to } \sum_{\theta<\theta_i} U(\theta)\mu_0(\theta) \geq \mathbf{q}(\theta)\sum_{\theta<\theta_i}\mu_0(\theta)(\theta_i-\theta) + \sum_{\theta<\theta_i} U(\theta_0)\mu_0(\theta), \text{ for } i \leq k \qquad \text{(D-IC}_i^{-1}\text{)}$$

$$\sum_{\theta>\theta_i} U(\theta)\mu_0(\theta) \geq \mathbf{q}(\theta) \sum_{\theta>\theta_i} \mu_0(\theta)(\theta - \theta_i) + \sum_{\theta>\theta_i} U(\theta_0)\mu_0(\theta), \text{ for } i > k \tag{D-IC$_i^1$}$$

$$U(\theta_i) \geq 0, \tag{IR$_i$}$$

That is, we get two independent problems: to the left and to the right of $\theta_k$ which splits the market into two locations. As verified in the proof of Theorem 1, in each of these problems, the appropriate surplus-based rationing achieves the optimum.

I now verify that the buyer is willing to communicate truthfully in the suggested extensive form. Given Theorem 1, it only remains to verify that truthful communication is incentive compatible after the first threshold $\theta_k$. By incentive-compatibility of bottom-up communication to the right of $\theta_k$, the type $\theta_{k+1}$ is now willing to deviate towards an allocation of a higher type:

$$\begin{aligned}\mathbb{E}[U(\theta_i)|i \leq k] \geq 0 = U(\theta_{k+1}) &\geq \max\{U(\theta_j) - q_j(\theta_j - \theta_k), 0\} \\ &\geq \mathbb{E}[\max\{U(\theta_j) - q_j(\theta_j - \theta_i), 0\}|i \leq k], \forall j \geq k+1\end{aligned}$$

which ensures that after the buyer learns he is below $\theta_k$, he has no incentives to misreport. By a symmetric argument, the buyer does not wish to misreport after learning his type is above $\theta_k$. This concludes the proof.

□