Markdowns

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Abstract

I model markdown pricing as a tool for indirect price discrimination by product value. New inventory of uncertain value is initially introduced at higher prices and unsold goods are gradually marked down. Consumers arrive sequentially, choose a price at which to inspect a good, observe its value, and decide whether to purchase. Since unsold goods are negatively selected, consumer search and purchase decisions endogenously sort products by their value across different markdown tiers. I introduce and characterize sorting equilibria: steady states in which consumer decisions and the distribution of inventory by its expected value across prices are mutually consistent. Despite complex sorting dynamics and a rich equilibrium set, the main result shows that equilibrium payoffs depend only on a single statistic: the gap in expected inventory value between the initial price tier and the terminal markdown tier. The paper also highlights a fundamental trade-off: greater sorting and price discrimination of goods comes at the cost of lower sales and reduced total welfare.

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1 Introduction

Many firms clear unsold inventories through markdowns. A vivid example is Filene's Basement, a Boston retailer that implemented an automated markdown system: if an item remained unsold for twelve days, its price was automatically reduced by 25%. After six more days, the markdown rose to 50%, and after another six days, to 75% of the original price (The New York Times, 1982). Today, similar strategies are widespread and take different forms: some firms move unsold goods to outlet locations, while others rely on clearance racks or "special offer" tags.

Lazear (1986) argues that markdowns allow sellers to price-discriminate by product quality,[1] even when that quality is observable only to consumers. Markdowns reflect prior demand: unsold goods are more likely to have lower consumer value. Even an automatic markdown system, such as that of Filene's, can therefore exploit the selection of unsold goods, without requiring the seller to track the performance of each individual item. This selection mechanism is especially valuable in settings where sellers face substantial marginal costs and high product variability but lack reliable prior demand data (*e.g.*, due to short life cycle of the product). Examples include apparel,[2] furniture, and toys.

In this paper, I endogenize consumer search and product sorting across price tiers. Due to search frictions, consumers care about the expected value of inventory at each price tier. At the same time, the expected inventory value is itself endogenous: products at later markdown tiers are negatively selected through consumer decisions.

The main result shows that even in the model with a continuum of markdown tiers, equilibrium consumer and seller surpluses are fully summarized by a single statistic: sorting precision. This statistic measures how much inventory's expected value deteriorates between the initial price tier and the terminal markdown tier. I apply this dimensionality reduction to study optimal markdowns for a monopoly seller. Seller-optimal sorting precision balances the inherent trade-off of markdown pricing: greater sorting precision reduces total surplus by lowering sales but lets the seller extract more of the consumer surplus.

Model. Durable goods are continuously produced and are offered at different price tiers. In the baseline model, the goods cannot be discarded and can only be sold (relaxed in Section 4.1). Some goods are more valuable to consumers, for example, because they are perceived as fashionable. For most of the analysis, product value is binary: each good is

[1]Throughout the paper, I refer to product quality as consumers' willingness to pay for the good rather than the objective characteristics of the good.

[2]For instance, Fisher and Raman (1996) provides a case study of Sport Obermeyer, a sportswear manufacturer that commits to production decisions about two years ahead, with 95% of its products being new designs.

either high- or low-value, and consumers are homogeneous.

The price tiers are indexed by locations on a segment between 0 and 1. Apart from prices, locations also differ in their inventory quality, measured by the share of high-value items. Depending on the exact markdown implementation, a "location" may correspond to a distinct store (flagship/outlet), a section within a store (front of the store/clearance rack), a price tag, or the product's vintage (see Section 5).

Consumers arrive sequentially at a constant rate and choose where to search for goods, given prices and inventory qualities. At the selected location, each consumer inspects one random good from its inventory, learns the value of the good, decides whether to purchase at the location's posted price, and exits the market. Goods are sorted through the negative selection: unsold inventory mechanically flows downstream ($0 \to 1$).

In a sorting equilibrium, consumer beliefs about inventory qualities become self-fulfilling. For example, consider a simplified environment with two locations: upstream high-priced flagship and a low-priced downstream outlet.[3] If consumers believe the outlet to have the same inventory quality as the flagship, they all strictly prefer to search at the outlet due to its lower price. As no consumers search at the flagship, they have no impact on its inventory. Inventory that flows to the outlet has mechanically the same quality as the flagship, and in the long run the beliefs of consumers get confirmed.

Panel (a) of Figure 1 summarizes a different, interior, equilibrium in the same example. Here, consumers are indifferent between locations: the flagship's higher price is compensated by its superior inventory quality. At the flagship's high price, only high-value items sell: its unsold inventory, flowing to the outlet, is negatively selected. In equilibrium, the flagship traffic adjusts to exactly sustain the suggested inventory qualities in the long run.

Main Results. The main results in Section 3 characterize all sorting equilibria. For arbitrary prices, Theorem 1 shows that every sorting equilibrium admits a threshold structure analogous to the flagship/outlet example. Locations upstream of the threshold act as *flagships*, charging higher, location-specific prices and selling only high-value items. Downstream of the threshold, consumers only search at low-priced locations and purchase both types of goods. Inventory quality gradually declines up to reaching the threshold location and remains constant thereafter.

Theorem 1 also delivers the irrelevance result. To compute equilibrium consumer or seller surplus, one only needs to evaluate the sorting precision between the initial price tier and the threshold location, while all other details of markdown or sorting paths are irrelevant. For the consumer surplus, the result is intuitive: it follows from the same indifference argument as in the two-location example. For the seller surplus, the result follows from the tight link

[3] That is, the outlet only contains the goods that are initially not purchased at the flagship.

between sales and sorting: the goods are sorted exactly through the lack of sales, so fixing any sorting rate mechanically fixes the sales rate (and revenue) at any location.

Proposition 1 provides comparative statics for the main equilibrium outcomes with respect to the sorting precision. The key takeaway is that markdown pricing necessarily entails a trade-off between sales volume and sorting. To illustrate, Panel (b) of Figure 1 plots another interior equilibrium, with a greater sorting precision. In this equilibrium, the flagship store has both higher price and consumer traffic so that the the negative selection on unsold inventory is stronger. As more consumers purchase selectively, the total sales fall, and the total welfare goes down.

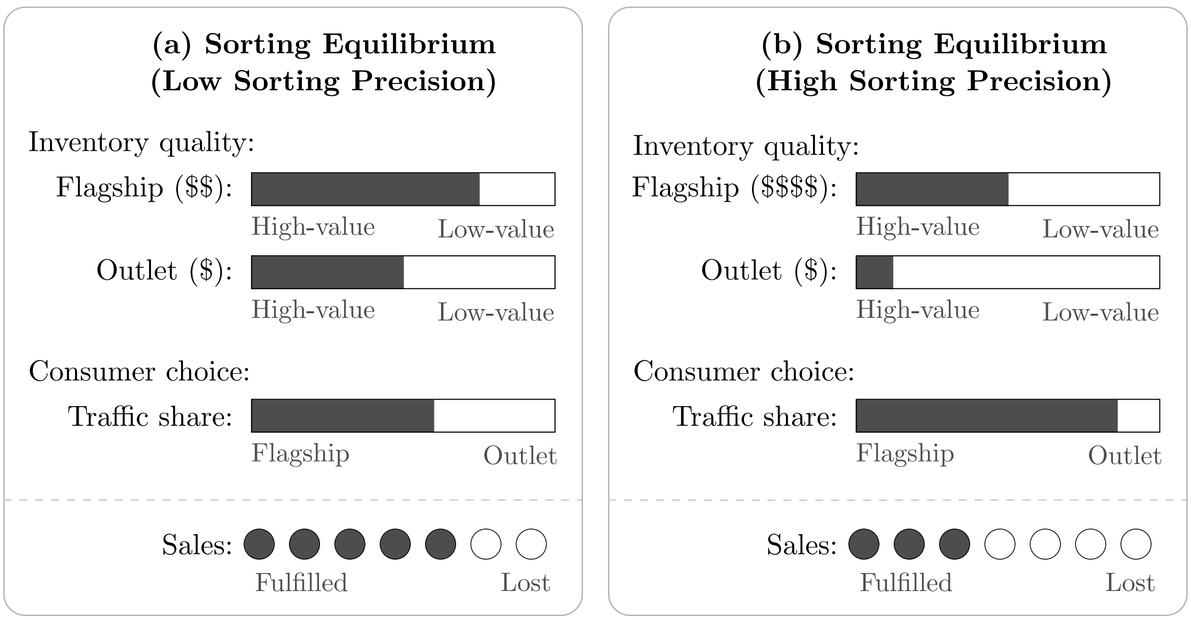


Figure 1: Sorting Equilibria in Flagship/Outlet Example

Note: The figure schematically depicts two interior sorting equilibria in a flagship/outlet example with different sorting precisions. The shaded area illustrates high-value inventory shares at each location (in inventory bars) and the share of consumers searching at the flagship (in traffic bars). Dollar signs indicate relative price level. Each panel reports sales as the key equilibrium outcome below the dashed line.

Application: Optimal Monopoly Pricing. Section 4 studies a pricing problem for a monopolistic seller. By the irrelevance result, seller's problem reduces to selecting the optimal level of sorting precision, which balances the trade-off between rent extraction and inefficiency, articulated in Proposition 1. The seller sorts products more in her optimum when high-value goods get more attractive to consumers, or when their share at the production gets (sufficiently) higher.

Extensions. The baseline model captures the equilibrium nature of markdowns in the most tractable setup. The paper also provides two extensions to the baseline model that add to its applicability.

First, the baseline model does not allow for direct disposal of goods, which is a common practice among many retailers. Section 6 extends model by letting the seller destroy her goods at a constant rate when they fail to sell even at a terminal markdown. Proposition 3 shows that in the seller's optimum, she uses only one disposal method for her low-value inventory: she either clears it at low-priced outlets, or destroys it directly to maintain high prices at all locations. When the product value is not binary, this result is no longer true: the seller can use markdowns to clear some of the medium-valued goods while only discarding the lowest-valued ones.

Second, the baseline model assumes homogeneous consumers, while existing literature often emphasizes markdowns as a price discrimination tool for different consumer types. Section 6.1 connects the two approaches by considering heterogeneous consumers, who differ in their preference for high-value items. As a result, we get consumer segmentation with menus as in the classical model by Mussa and Rosen (1978). The key difference is that the menus are not directly chosen by the seller but emerge endogenously through the same sorting mechanism as in the baseline model.

2 Model

This section describes the model of equilibrium markdowns with a rich pricing structure. The goods have uncertain consumer value and are offered at a continuum of locations that differ in their posted prices and inventory qualities. Consumers choose where to search for goods, draw one good at random, learn its quality, and decide whether to purchase. I introduce a sorting equilibrium, which formalizes endogenous product sorting by consumer value through consumer choice. It imposes steady-state restrictions on prices, consumer behavior, and the quality distribution.

Products. The goods are durable and are differentiated by consumer value. For now, the product's value is binary and can be either high or low.[4] Consumers (males) derive utility v^h from high-value products and v^l from low-value ones, where $v^h > v^l > 0$. All goods have identical marginal cost that I normalize to 0.[5]

Locations. The goods are offered over a continuum of *locations* indexed by $x \in X = (0, 1)$. Location 0 is the production plant. The total inventory is normalized to 1 and is uniformly

[4] I discuss how this assumption can be relaxed in Section 6.3.

[5] I could alternatively shift consumer values by the constant marginal cost of production. I elaborate on this later, in Section 4.1.

distributed over X.[6] Each location $x \in X$ is characterized by its price $\mathbf{p}(x)$ and its inventory quality $\mathbf{q}(x)$, *i.e.* the share of high-quality products in its stock. Both the *price schedule* $\mathbf{p} : X \to \mathbb{R}$ and the *inventory quality* $\mathbf{q} : [0, 1) \to [0, 1]$ are (Lebesgue-)measurable. The probability of a high-value product at the production plant, $\mathbf{q}(0)$, is fixed at $\pi \in (0, 1)$.

Consumers. Time is continuous, and at every instant, a flow of short-lived consumers arrives at the market at a unit rate. They choose how to allocate limited attention across X.[7] The *consumer strategy* is a density function $\sigma : X \to \mathbb{R}_+$ that describes the distribution of consumer attention.[8]

Each consumer inspects a single product at random. The consumer strategy determines where the inspected product is drawn: the more attention a consumer pays to a given set of locations, the more likely he is to draw a product from there.[9] A location x is *visited* if it gets positive attention from consumers, *i.e.,* $\sigma(x) > 0$. Throughout, I say that a condition A is satisfied on an interval $[x_1, x_2] \subseteq [0, 1]$ σ-*almost surely (a.s.)* if the measure of visited locations that fail A is zero: $\int_{x \in [x_1, x_2]} \mathbb{1}\{x \text{ fails } A\}\sigma(x)dx = 0$.

Conditional on the location, a product is drawn at random from its available inventory. If location x holds a share $\mathbf{q}(x)$ of high-value goods, then the consumer finds a high-value item at x with probability $\mathbf{q}(x)$. Upon inspecting the drawn product at location x, the consumer learns its value and decides whether to buy it at price $\mathbf{p}(x)$.[10] The consumer earns a payoff $v^\omega - p$ when purchasing a product of type $\omega \in \{l, h\}$ at price p.

Prices determine which qualities are purchased at each location. A product of type ω is purchased at location x whenever $v^\omega \geq \mathbf{p}(x)$. I refer to locations that sell both types of goods as outlets. Formally, a location x is *an outlet* if $\mathbf{p}(x) \leq v^l$. Otherwise, it is a *non-outlet* location.

Expected consumer payoff depends on a *market outcome* $m = (\mathbf{p}, \sigma, \mathbf{q})$: a tuple of the price schedule, the consumer strategy, and the inventory quality. To summarize, consumer strategy σ determines where the consumer inspects, inventory quality $\mathbf{q}$ then determines the odds of finding a high-value good, and the price $\mathbf{p}$ sets the terms of trade. The expected

[6] For the subsequent analysis, it is enough to assume that the distribution of stock is absolutely continuous with respect to Lebesgue measure on X. The uniform distribution is a normalization.

[7] Equivalently, consumers choose a single location to visit. Consumer strategy is then simply their mixing strategy over all locations.

[8] In Section 6.2, I relax the assumption that consumer attention strategy is absolutely continuous and admits a density.

[9] For instance, the probability of drawing a product from an interval $[x_1, x_2]$ is $\int_{x_1}^{x_2} \sigma(y)dy$.

[10] Alternatively, the consumer learns the value after the purchase but can return the good to the same location at no cost.

consumer payoff at the market outcome m is:[11]

$$V^B(m) \triangleq \int_{x \in X} \left[\mathbf{q}(x)(v^h - \mathbf{p}(x))_+ + (1 - \mathbf{q}(x))(v^l - \mathbf{p}(x))_+ \right] \sigma(x) dx.$$

Pricing Process: Product Flows. I now model how the goods get sorted through the lack of sales. Unsold inventory flows linearly in the same direction from 0 to 1: each location receives inventory from its immediate upstream neighbor and passes goods downstream.[12] For now, all inventory is eventually purchased and cannot be discarded (I relax this assumption in Section 4.1). Note that as locations have different posted prices, every instant the good is reallocated downstream, it gets repriced.

The rate of product flows explicitly links sales to selection of goods that get repriced. In a market outcome m, unsold inventory is passed downstream of location x at the rate of its *downstream sales* $S_m(x)$:[13]

$$S_m(x) \triangleq \int_{y \in (x,1)} \left[\mathbf{q}(y)\sigma(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\} + (1 - \mathbf{q}(y))\sigma(y)\mathbb{1}\{\mathbf{p}(y) \leq v^l\} \right] dy.$$

Say that a location x is *a transition location* in a market outcome m if it passes a positive flow of goods: $S_m(x) > 0$. I assume that the reallocated inventory is picked at random: high-value products flow through the location x at a rate $\mathbf{q}(x)S_m(x)$.

Figure 2 summarizes all inventory flows over a fixed interval of locations $(x_1, x_2] \subseteq X$. First, inventory exits this interval due to consumer purchases. The total mass of high-value purchases within the interval is:

$$\int_{y \in (x_1, x_2]} \mathbf{q}(y)\sigma(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\} dy,$$

and the total mass of low-value purchases is:

$$\int_{y \in (x_1, x_2]} (1 - \mathbf{q}(y))\sigma(y)\mathbb{1}\{\mathbf{p}(y) \leq v^l\} dy.$$

In addition to purchases, inventory exits the interval through its right boundary x_2, which passes its products to downstream locations.[14] The total inventory mass reallocated downstream from x_2 is $S_m(x_2)$, of which mass $\mathbf{q}(x_2)S_m(x_2)$ is high-value. The interval receives

[11] Here, and elsewhere $(a)_+ \equiv \max\{a, 0\}$ for any $a \in \mathbb{R}$.

[12] For any two locations, $x < y$, location is y is downstream of x and location x is upstream of y.

[13] Given linear product flows, these rates ensure each location is kept at its full capacity at all times.

[14] Interior locations (x_1, x_2) also pass goods downstream, but these shipments only redistribute inventory within the interval $(x_1, x_2]$ and do not affect its inflows or outflows.

inventory exclusively from location x_1. The total inflow is $S_m(x_1)$, and the inflow of high-value goods is $\mathbf{q}(x_1)S_m(x_1)$.

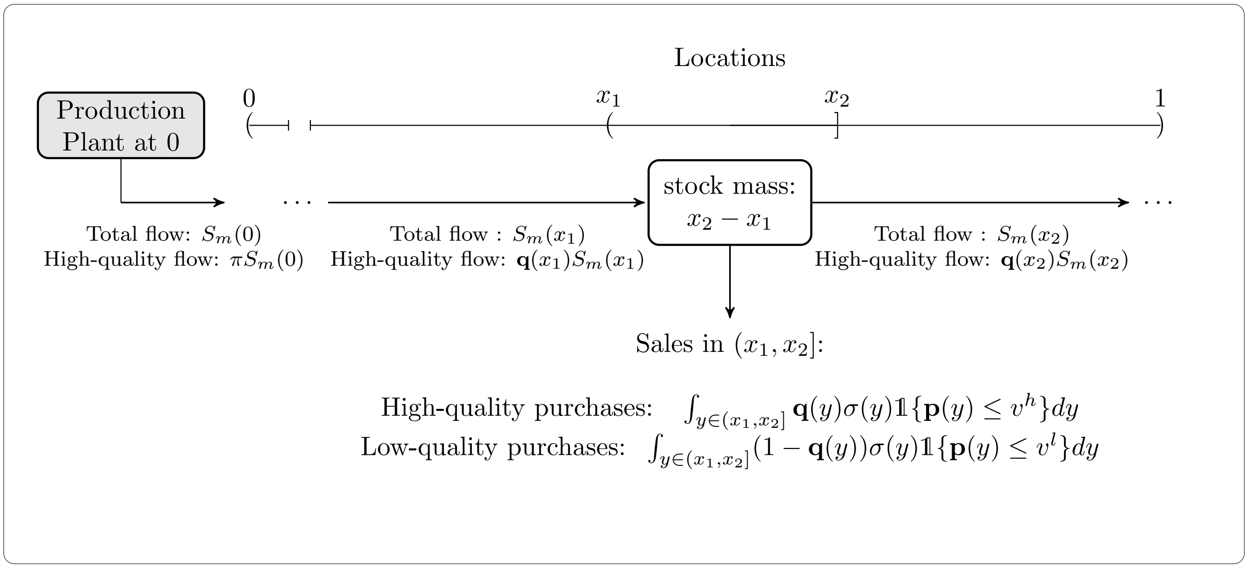


Figure 2: Product Flows within a Period

The mass of inventory is kept constant by construction, but the average inventory quality within an interval $(x_1, x_2]$ may change over time. It only stays constant when outflows and inflows of high-value goods are balanced:

$$\int_{y\in(x_1,x_2]} \mathbf{q}(y)\sigma(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\}dy + \mathbf{q}(x_2)S_m(x_2) = \mathbf{q}(x_1)S_m(x_1). \tag{1}$$

I say that the inventory quality $\mathbf{q}$ is *sustained on* $A \subseteq X$ by prices and consumer strategy $(\mathbf{p}, \sigma)$ if Equation (1) holds for any interval $(x_1, x_2] \subseteq A$. For brevity, say that $\mathbf{q}$ is *sustained* by $(\mathbf{p}, \sigma)$ if it is sustained on X.

Sorting Equilibrium. The central goal of the model is to capture the interdependence between prices, consumer choices, and inventory quality. To capture it in a tractable way, I use an equilibrium concept that imposes steady-state restrictions on the market outcomes. A market outcome $(\mathbf{p}, \sigma, \mathbf{q})$ is *a sorting equilibrium* if:

1. the inventory quality $\mathbf{q}$ is sustained by $(\mathbf{p}, \sigma)$;

2. each visited location maximizes consumer payoff given $(\mathbf{p}, \mathbf{q})$:

$$x \in \arg\max_{y\in X} \mathbf{q}(y)(v^h - \mathbf{p}(y))_+ + (1-\mathbf{q}(y))(v^l - \mathbf{p}(y))_+.$$

Let $\mathcal{E}$ denote the set of sorting equilibria.

The sorting equilibrium captures how consumer beliefs about the inventory quality become self-fulfilling. Suppose consumers follow a time-invariant strategy σ, then the inventory quality at every period t is described by:[15]

$$\begin{aligned} \partial_t \mathbf{q}_t(x) &= -\sigma(x)\mathbf{q}_t(x)\mathbb{1}\{\mathbf{p}(x) \leq v^h\} - \partial_x \left[S_{m_t}(x)\mathbf{q}_t(x) \right], \\ \mathbf{q}_t(0) &= \pi. \end{aligned}$$

A sustained inventory quality $\mathbf{q}$ is a time-invariant solution to the above, capturing the long-run limit.[16] To justify a time-invariant consumer strategy, assume consumers do not observe either the calendar date or the current inventory quality. Their beliefs are dominated by the long-run limit of inventory quality.

It is worth emphasizing that the sorting equilibrium does not articulate how the prices are set, and only captures how consumers and products get sorted across locations in a mutually-consistent way. As I show in Section 3, despite this price flexibility, the sorting equilibrium imposes very sharp structure on most economic outcomes.

Efficient Benchmark. The total surplus of a market outcome $m = (\mathbf{p}, \sigma, \mathbf{q})$ equals the expected value of purchased goods:

$$TS(m) \triangleq \int_{y \in X : \mathbf{p}(y) \in (v^l, v^h]} v^h dS_m(y) + \int_{y \in X : \mathbf{p}(y) \leq v^l} \mathbf{q}(y)v^h + (1 - \mathbf{q}(y))v^l dS_m(y).$$

A market outcome is *efficient* if its total surplus equals $\pi v^h + (1 - \pi)v^l$, the average value of a new good.

Discussion. The model aims to capture three key features of markdown pricing. First, the selection of remaining inventory is leveraged to indirectly price-discriminate goods by their consumer value. Second, consumers choose prices endogenously, and these choices determine the strength of selective pressure on unsold goods. Third, selling opportunities are scarce, and there is an exploration/exploitation trade-off. The assumptions discussed below are chosen to isolate these forces in a rich but tractable environment.

The suggested pricing process makes the connection between sales and sorting most transparent. Furthermore, as discussed in Section 5, this location-based pricing process yields the same economic predictions as a vintage-based pricing model, in which goods are deterministically repriced as they age. With vintage pricing, linear product flows arise naturally and closely resemble the markdown strategies used in practice, including automatic markdowns

[15] Where $m_t = (\mathbf{p}, \sigma, \mathbf{q}_t)$

[16] To make this argument crisper, one would need to show convergence of $\mathbf{q}_t$ to the steady state. In Section Appendix C, I show convergence in simulations. In Online Section OA1, I verify convergence for a simpler version of the model with two stores used for the numerical example.

of Filene's Basement. I begin with the location-based formulation because its equilibrium concept is simpler (distribution of inventory volumes across locations is exogenous).

The assumption that each consumer inspects a single good from one location ensures that selling opportunities are scarce, and consumer choice is tractable. The main driving forces of the model would be the same if consumer could instead make multiple (but finite) draws. An equivalent interpretation of this assumption is that the seller faces a sequence of short-lived selling opportunities, capturing purchase impulses or transient demand shocks.

Finally, the analysis focuses on steady states. This allows me to capture dynamic sorting and pricing processes in a tractable static environment. An alternative simplification would be to consider a finite-horizon pricing model for a fixed inventory stock. Such a model can capture selection of unsold goods, but it cannot generate the exploration/exploitation trade-off. In a steady-state model, retaining inventory at high prices improves selection of remaining inventory but is costly as fewer new goods are introduced.

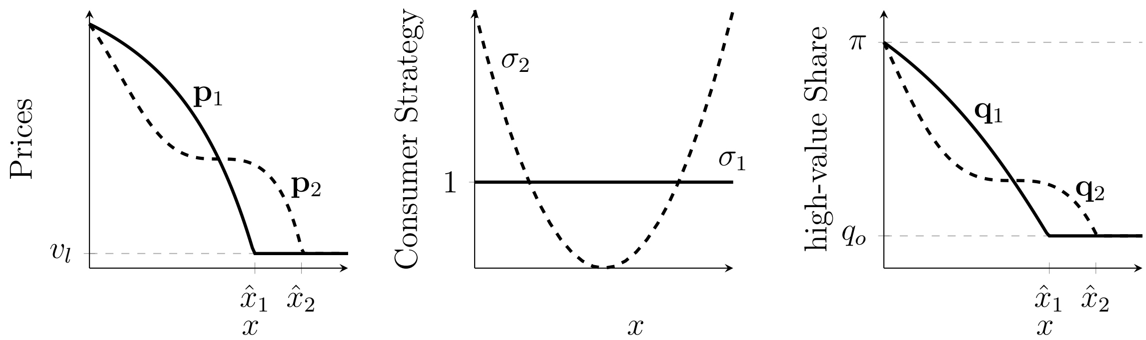
3 Sorting Equilibria

This section characterizes all sorting equilibria. In general, the set of sorting equilibria is very rich, due to both flexible prices and strategic complementarities in consumer attention choice. However, Theorem 1 establishes two properties that greatly simplify equilibrium analysis.

First, every sorting equilibrium has a threshold structure separating the visited locations into outlets and non-outlets. Second, in every equilibrium with non-trivial pricing, total surplus and consumer payoff depend only on the inventory quality at this threshold, or equivalently on the overall sorting precision. Proposition 1 relates sorting precision to sales, prices, and consumer traffic.

3.1 Characterization

Figure 3 illustrates the essence of Theorem 1 by plotting two different sorting equilibria. In each equilibrium, locations downstream of the threshold $\hat{x}_i$, $[\hat{x}_i, 1]$ are outlets: prices are low, purchases are value-neutral, and inventory quality remains constant. Locations upstream of $\hat{x}_i$, are non-outlets that only sell high-value goods. There, unsold inventory becomes increasingly negatively selected, so that the inventory quality $\mathbf{q}_i$ decreases over $[0, \hat{x}_i]$.



(a) Prices *(b) Consumer Strategy* *(c) Inventory Quality*

Figure 3: Two Examples of the Sorting Equilibrium

Note: the figure illustrates two sorting equilibria $(\mathbf{p}_1, \sigma_1, \mathbf{q}_1)$ (solid lines) and $(\mathbf{p}_2, \sigma_2, \mathbf{q}_2)$ (dashed lines). The two equilibria are different in their paths of pricing and sorting, but have the same overall sorting precision of π/q_o.

Theorem 1 shows that this threshold structure is a general equilibrium feature and establishes the irrelevance result: equilibrium payoffs only depend on the inventory quality at the outlet threshold. In particular, the two plotted equlibria, despite having very distinct shapes, are payoff-equivalent.

To state the result, I provide some additional definitions. Formally, I say every market outcome $m = (\mathbf{p}, \sigma, \mathbf{q})$ admits *an outlet threshold* $\hat{x}$ if $\mathbf{p}(\cdot) > v^l$ on $(0, \hat{x})$ (σ-a.s.), and $\mathbf{p}(\cdot) \leq v^l$ on $[\hat{x}, 1)$. I say that a sorting equilibrium is *neutral* if $\mathbf{q}(\hat{x}) = \pi$ and I say it is *active* if $\mathbf{q}(\hat{x}) \in (0, \pi)$.

Theorem 1. *Suppose a market outcome $m = (\mathbf{p}, \sigma, \mathbf{q})$ is a sorting equilibrium. Then, it admits an outlet threshold for some $\hat{x} \in [0, 1]$. Furthermore:*

(i) If no consumers visit outlet locations, i.e., $\int_{\hat{x}}^1 \sigma(y)dy = 0$, then both consumer and seller payoffs are zero.

(ii) If (almost) all consumers visit outlet locations, i.e., $\int_{\hat{x}}^1 \sigma(y)dy = 1$, then the sorting equilibrium is neutral, and efficient, i.e., $TS(m) = \pi v^h + (1 - \pi)v^l$. Consumer payoff is at least $\pi(v^h - v^l)$.

(iii) If consumers visit both outlet and non-outlet locations, i.e., $\int_{\hat{x}}^1 \sigma(y)dy \in (0, 1)$, then the sorting equilibrium is active, and inefficient. The payoffs are fully determined by the inventory quality at the outlet threshold $\hat{x}$, with:

$$TS(m) = \frac{\pi v^h + (1 - \pi)v^l}{\ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1 - \pi) + 1},$$

$$V^B(m) = \mathbf{q}(\hat{x})(v^h - v^l).$$

Finally, for any $q \in (0, \pi]$*, there exists a sorting equilibrium* $(\mathbf{p}, \sigma, \mathbf{q})$ *with positive sales that has inventory quality* q *at the outlet threshold* $\hat{x}$*:* $\mathbf{q}(\hat{x}) = q$*.*

Theorem 1 divides all sorting equilibria into three types, based on how consumers distribute their attention between outlet and non-outlet locations. Only the last, interior, case features active sorting while having positive total sales. The payoffs in this case depend only on the inventory quality at the outlet threshold $\mathbf{q}(\hat{x})$, rendering other equilibrium details payoff-irrelevant.

The irrelevance result formulated in part (*iii*) can be equivalently stated in terms of sorting precision. For any equilibrium with an outlet threshold, define *sorting precision* as $\pi/\mathbf{q}(\hat{x})$. This definition extends directly from the flagship/outlet example in the introduction.[17] As the main focus of the paper is to understand the sorting of goods through the lack of sales, I interpret most of my results using the sorting precision.

Even though the set of sorting equilibria is infinitely dimensional, Theorem 1 implies the set of attainable payoffs is one-dimensional. For every market outcome m, the seller surplus $V^S(m)$ is just the difference between the total surplus and consumer payoff:

$$V^S(m) \triangleq TS(m) - V^B(m).$$

Figure 4 plots all attainable pairs of consumer and seller surpluses (V^B, V^S). Greater sorting precision shrinks total surplus but reallocates more of it to the seller(s).

[17] There, it is the ratio between the inventory quality at the flagship and the outlet. In the continuous model, since the earliest visited location has inventory quality π, the sorting precision reduces to $\pi/\mathbf{q}(\hat{x})$.

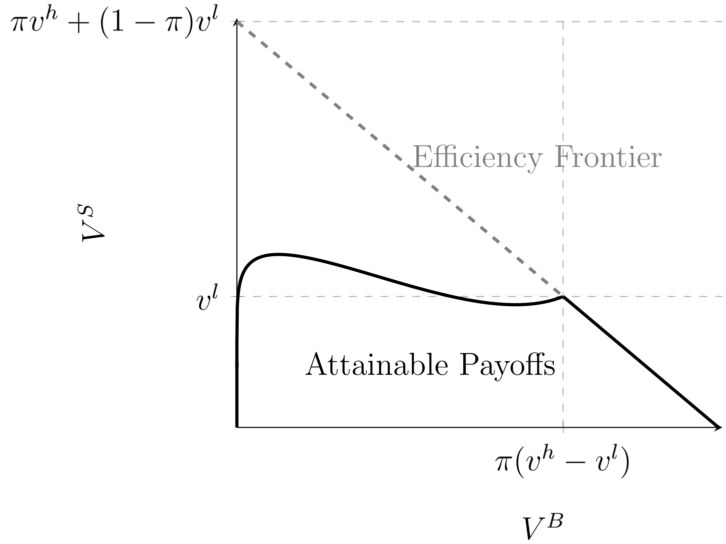


Figure 4: Attainable Equilibrium Payoffs

Equilibrium Comparative Statics. Given the irrelevance result, Proposition 1 summarizes how the key equilibrium market observables change with the sorting precision. It mirrors the main insights of the flagship/outlet example. To account for many different prices, Proposition 1 focuses on the *average non-outlet transaction price* in an equilibrium market outcome $m = (\mathbf{p}, \sigma, \mathbf{q})$ with an outlet threshold $\hat{x}$:

$$\frac{\int_0^{\hat{x}} \mathbf{p}(x)\mathbf{q}(x)\sigma(x)dx}{\int_0^{\hat{x}} \mathbf{q}(x)\sigma(x)dx}.$$

Similarly, the *average searched non-outlet quality* at m is defined as:

$$\frac{\int_0^{\hat{x}} \mathbf{q}(x)\sigma(x)dx}{\int_0^{\hat{x}} \sigma(x)dx}.$$

Proposition 1. *Consider two sorting equilibria $m_1 = (p_1, \sigma_1, \mathbf{q}_1)$ and $m_2 = (p_2, \sigma_2, \mathbf{q}_2)$ with outlet-thresholds $\hat{x}_1, \hat{x}_2$, respectively. Suppose that at both equilibria, consumers visit both outlet and non-outlet locations. Then, the following are equivalent:*

(i) m_1 has a greater sorting precision than m_2, i.e., $\mathbf{q}_1(\hat{x}_1) < \mathbf{q}_2(\hat{x}_2)$,

(ii) m_1 has lower total sales than m_2, i.e., $S_{m_1}(0) < S_{m_2}(0)$,

(iii) m_1 has greater non-outlet consumer share than m_2, i.e. $\int_0^{\hat{x}_1} \sigma_1(x)dx > \int_0^{\hat{x}_2} \sigma_2(x)dx$,

(iv) the average non-outlet transaction price is higher at m_1 than at m_2,

(v) the average searched non-outlet quality is lower at m_1 than at m_2,

(vi) both total surplus and consumer surplus are lower at m_1 than at m_2.

Proof. See Appendix C. □

Equivalence between parts (*i*) - (*ii*) formalizes the key trade-off in markdown pricing: sorting of products always comes at the cost of lower sales volume. This result is intuitive given the products are sorted through the lack of sales.

Figure 5 illustrates the two sources of sales losses. The first effect is direct: a higher sorting precision requires more consumers to search at high prices, so that the negative selection of unsold inventory is stronger (part (*iii*) of Proposition 1). These consumers do not purchase low-value items when they find them, in contrast consumers who draw their goods at outlets. The second effect is due to lower turnover rate, which worsens inventory quality at non-outlet locations (part (*iv*) of Proposition 1). Consumers searching at these locations are more likely to find low-value items, that they do not purchase.

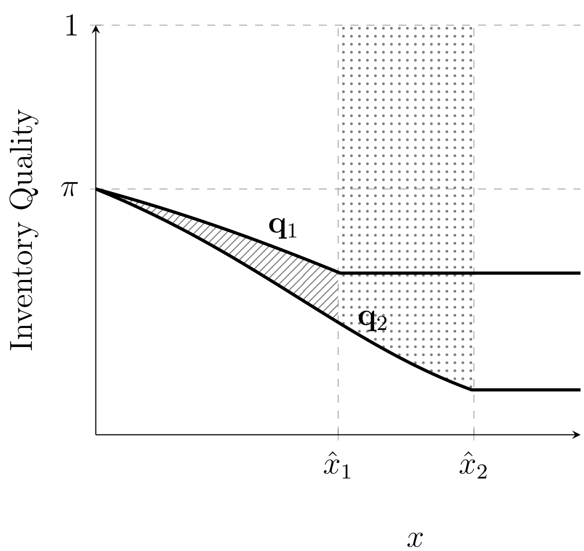


Figure 5: Two Sources of Sales Loss

Note: The figure compares two sorting equilibria with different sorting precisions. The two equilibria have the same consumer strategy $\sigma = 1$, but different inventory quality and prices. By decreasing the share of outlet attention (moving from $\hat{x}_1$ to $\hat{x}_2$), the sorting precision increases. The inventory quality worsens across all locations. The shaded area shows the resulting total sales loss: the dotted hatch represents the direct effect (fewer purchases at higher prices), and the diagonal hatch captures the quality-composition effect (slower inventory turnover).

Note that the equivalence between parts (*iii*) and (*iv*) implies a positive relationship between non-outlets' prices and their customer shares, resembling a Veblen effect. In my model, it is not driven by any assumptions on consumer preferences but arises endogenously in equilibrium. Intuitively, when more consumers shop at non-outlet locations, the outside option of outlets worsens. This acts as a negative income effect, increasing the "demand" for non-outlets.

3.2 Proof Sketch of Theorem 1

The remainder of the section outlines the main steps in the proof of Theorem 1 and illustrates how sorting equilibria are constructed. Formal proofs and technical details are deferred to the Appendices.

Sorting Process. The starting point is to understand how prices and consumer strategies shape inventory qualities at all locations. Lemma 1 summarizes the key restrictions on $\mathbf{q}$ that can be sustained over a given interval by some prices and consumer strategy $(\mathbf{p}, \sigma)$. It considers two cases, depending on whether the visited locations in this interval are outlets.

Lemma 1. *Consider some market outcome* $m = (\mathbf{p}, \sigma, \mathbf{q})$.

(i) Suppose that $\mathbf{p}(\cdot) > v^l$ *(σ-a.s.) over* $[x_1, x_2] \subseteq [0, 1)$*. Then,* $\mathbf{q}$ *is sustained by* $(\mathbf{p}, \sigma)$ *on* $[x_1, x_2]$ *if and only if* $S_m(x)(1 - \mathbf{q}(x))$ *is constant over* $[x_1, x_2]$*.*

(ii) Suppose that $\mathbf{p}(\cdot) \leq v^l$ *(σ-a.s.) over* $[x_1, x_2] \subseteq [0, 1)$ *and* x_2 *is a transit location. Then,* $\mathbf{q}$ *is sustained by* $(\mathbf{p}, \sigma)$ *on* $[x_1, x_2]$ *if and only if* $\mathbf{q}(x)$ *is constant over* $[x_1, x_2]$*.*

Proof. See Appendix C for a proof. □

In words, Lemma 1 part (*i*) states that on any interval of non-outlet locations, low-value inventory is passed downstream at a constant rate. This is because low-value goods are not purchased there. Part (*ii*) states that over any interval of outlet locations, the inventory quality stays constant. Intuitively, the purchases are value-neutral, so there is no selection on the remaining inventory that gets passed downstream.

Threshold Structure. Intuitively, the the threshold structure of any equilibrium sorting outcome follows from the inventory quality $\mathbf{q}(\cdot)$ being non-increasing. To elaborate, suppose consumers visit some outlet location x and let y be the first visited non-outlet location downstream of x,*i.e.,* locations in $[x, y]$ are outlets (σ-a.s.). Lemma 1 part (*ii*) implies the inventory quality of x equals that of y, while the price at y is strictly higher. Then the location x delivers a strictly higher payoff than y, so that y must not be visited in an equilibrium.

No Outlets: Sales Collapse. If no outlets are visited, the sales collapse. In this case, all locations are non-outlets ($\sigma-$a.s.). and by Lemma 1 part (i), we must have $(1-\pi)S_m(0) = (1-\mathbf{q}(1-))S_m(1-) = 0$. As there are no sales downstream of 1, *i.e.,* $S_m(1-) = 0$, this condition implies the total sales $S_m(0)$ to be zero. Intuitively, in the absence of low-priced sales, the low-quality items gradually fill all available shelf space, crowding out any purchases at prices above v^l.

Only Outlets: Neutral Equilibrium. Moving to part (ii) of the theorem, assume that all visited locations are outlets. Then, the seller charges at most v^l from all consumers. From Lemma 1 part (ii), the goods do not get sorted: the inventory quality is constant over all (transit) locations and equals π, the average quality at the production plant. All consumers make a purchase, and the sorting equilibrium is efficient.

Interior Case: Active Sorting and Irrelevance. Let us now consider the case from part (iii), where consumers visit both outlet and non-outlet locations. From the threshold structure and Lemma 1, the inventory quality is continuous and decreasing on $[0, \hat{x}]$ (because $S_m(\cdot)$ is continuous and decreasing) and is constant on $[\hat{x}, 1]$.

Consumer Payoff. If consumers visit both location types, their payoff is $V^B(m) = \mathbf{q}(\hat{x})(v^h - v^l)$. This follows from consumer indifference between all visited locations and the properties of the inventory quality from above. By assumption, some outlet locations, which all have inventory quality $\mathbf{q}(\hat{x})$, are visited. Then, by indifference, consumer payoff is at least $\mathbf{q}(\hat{x})(v^h - v^l)$. For any non-outlet location y, the payoff is $\mathbf{q}(y)(v^h - \mathbf{p}(y))$, which is at most $\mathbf{q}(y)(v^h - v^l)$, since the price at y is strictly above v^l. By continuity of $\mathbf{q}(\cdot)$, there is some visited non-outlet location y with the inventory quality, that is arbitrarily close to $\mathbf{q}(\hat{x})$. If any visited outlet charges a price strictly below v^l, it delivers a strictly higher payoff than y, contradicting consumer indifference. Hence, all visited outlets must charge v^l, and $V^B(m) = \mathbf{q}(\hat{x})(v^h - v^l)$.

Total Surplus. The inventory quality at the outlet threshold determines the total surplus in a sorting equilibrium. The main driving force behind this result is that sales generate both surplus and sorting. As goods are sorted through the lack of sales, any sorting rate necessarily fixes the sales rate. In turn, this fixes the rate at which the market generates trade surplus.

First, Lemma 2 shows that the total surplus in a sorting equilibrium is as if only new products are drawn, corrected for the actual sales volume.

Lemma 2. *If $m = (\mathbf{p}, \sigma, \mathbf{q})$ is a sorting equilibrium, then the total surplus at m is given by:*

$$TS(\mathbf{p}, \sigma, \mathbf{q}) = \left(\pi v^h + (1-\pi)v^l\right) S_m(0). \tag{2}$$

Proof. To prove Lemma 2, consider the total surplus generated at outlet and non-outlet locations separately. At outlet locations $[\hat{x}, 1)$, inventory quality is constant at $\mathbf{q}(\hat{x})$ (σ-a.s.) by Lemma 1, and consumers buy both product types. Thus, the total surplus generated at outlets equals:

$$\left[\mathbf{q}(\hat{x})v^h + (1 - \mathbf{q}(\hat{x}))v^l\right] S_m(\hat{x}).$$

At non-outlet locations $(0, \hat{x})$, consumers only purchase high-value goods. Thus, they generate the surplus of

$$v^h(S_m(0) - S_m(\hat{x})).$$

Summing these, we obtain that the total surplus generated on X is:

$$\begin{aligned} TS(\mathbf{p}, \sigma, \mathbf{q}) &= v^h(S_m(0) - S_m(\hat{x})) + \left[\mathbf{q}(\hat{x})v^h + (1 - \mathbf{q}(\hat{x}))v^l\right] S_m(\hat{x}) \\ &= v^h S_m(0) - (v^h - v^l)(1 - \mathbf{q}(\hat{x}))S_m(\hat{x}). \end{aligned}$$

Finally, using Lemma 1, we may replace $(1 - \mathbf{q}(\hat{x}))S_m(\hat{x})$ in the above with $(1 - \pi)S_m(0)$, which completes the proof of the lemma. □

Second, Lemma 3 establishes the identity, which relates the sorting precision to the sales volume in any sorting equilibrium.

Lemma 3. *Consider a $\hat{x}$-threshold market outcome $m = (\mathbf{q}, \sigma, \mathbf{p})$ with positive total sales $S_m(0)$. If m is a sorting equilibrium, then:*

$$S_m(0)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1-\pi) + 1\right] = 1. \tag{Q-S}$$

Proof. See Appendix C. □

4 Optimal Monopoly Pricing

In this section I solve for optimal monopoly pricing. By Theorem 1, equilibrium payoffs only depend on the inventory quality at the outlet threshold. The seller's pricing problem therefore reduces to selecting this single inventory quality. At the optimum, the seller balances the trade-off between sales volume and consumer surplus extraction.

Suppose all locations are managed by a single seller setting prices at all locations. The seller's profit in a market outcome m thus equals the total seller surplus, $V^S(m)$. For any price schedule, I assume that nature selects the seller-preferred sorting equilibrium: the seller can nudge and coordinate consumers towards her preferred self-sustained beliefs about inventory qualities (*e.g.*, through a marketing campaign). We can thus define the seller's problem as selecting a sorting equilibrium directly:

$$\sup_{m \in \mathcal{E}} V^S(m).$$

The irrelevance result of Theorem 1 implies Corollary 1, which reduces the seller's problem to choosing the inventory quality at the outlet threshold.

Corollary 1. *The seller's problem is equivalent to choosing the inventory quality at the outlet threshold in the following sense. Given parameters (π, v^h), a sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q})$ with an outlet threshold $\hat{x}$ is a solution to the seller's problem if and only if*

$$\mathbf{q}(\hat{x}) \in \underset{q \in [0,\pi]}{\mathit{argmax}}\, \tilde{V}^S(q, \pi, v^h),$$

$$\textit{where } \tilde{V}^S(q, \pi, v^h) \triangleq \begin{cases} \dfrac{\pi v^h + (1-\pi) v^l}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)(1-\pi) + 1} - q(v^h - v^l), & \textit{if } q \in (0, \pi], \\ \\ 0, & \textit{if } q = 0, \end{cases}$$

Proposition 2 characterizes the solution to the seller's problem and shows how the seller-optimal level of sorting changes with the model's parameters.

Proposition 2. *There exist thresholds $\bar{\pi}(v^h) \in (0,1)$ and $\bar{v}^h(\pi) \in (v^l, \infty)$ such that neutral sorting is optimal for the seller, i.e. $\pi \in \underset{q \in [0,\pi]}{\mathit{argmax}}\, \tilde{V}^S(q, \pi, v^h)$, if and only if $\pi \leq \bar{\pi}(v^h)$ or $v^h \leq \bar{v}^h(\pi)$. Otherwise, active sorting is optimal and $\underset{q \in [0,\pi]}{\mathit{argmax}}\, \tilde{V}^S(q, \pi, v^h) = q^o(\pi, v^h)$ for some $q^o(\pi, v^h) \in (0, \pi)$, where $q^o(\pi, v^h)$*

(i) decreases in v^h, and

(ii) decreases in π for $\pi \geq \hat{\pi}(v^h)$ for some $\hat{\pi}(v^h) < 1$.

Proof. See Appendix F. □

Proposition 2 states that the seller sorts products more when her expected benefits from price discrimination increase. In particular, when either π or v^h is sufficiently low, the seller attains her optimal profit with neutral sorting. In this case, she does not attempt any price discrimination of goods: prices are constant over time and equal to v^l. Otherwise, the seller strictly benefits from active sorting and dynamic markdowns.

When active sorting is optimal, seller-optimal inventory quality at the outlet threshold $q^o(\pi, v^h)$ balances the trade-off between sales and sorting. Figure 6 illustrates the costs and benefits of higher sorting precision for the seller. When the sorting precision increases, the seller benefits by extracting more of the consumer surplus but bears the inefficiency cost due to losing more of the potential sales:

$$-\partial_q \tilde{V}^S(q) = \overbrace{-\frac{\pi v^h + (1-\pi)v^l}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}\frac{1-\pi}{(1-q)q}}^{\text{Sorting Inefficiency due to Sales Loss}} + \underbrace{(v^h - v^l)}_{\text{CS Extraction}}.$$

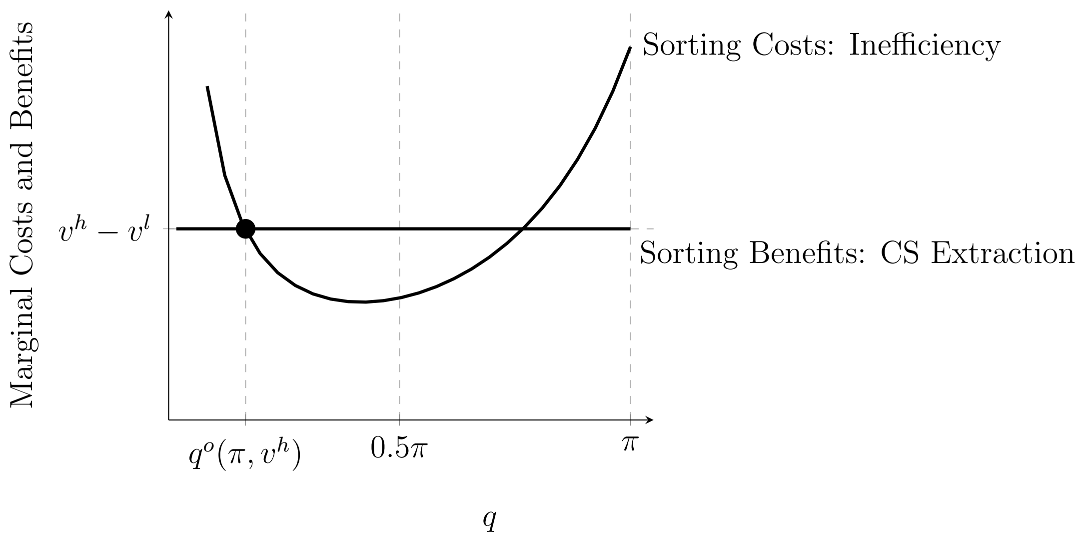


Figure 6: Marginal Effects of Higher Sorting Precision

Note: the figure captures the marginal effects of product sorting. As the sorting precision rises (q decreases), the seller benefits by capturing more consumer surplus. At the same time, the seller bears the sorting costs due to inefficiency from foregone sales. $q^o(\pi, v^h)$ denotes the optimal outlet inventory quality when active sorting is preferred.

4.1 Direct Disposal Extension

This section extends the baseline model by allowing for direct disposal of unsold inventory. This extension captures a realistic alternative to markdowns for clearing low-value goods. In the optimum, the seller uses only one channel to clear low-value goods from the stock: disposal and markdowns are not used simultaneously. The extension also lays the groundwork for comparing the baseline model to an alternative, vintage-based, markdown process in Section 5.

Assume the seller may destroy unsold inventory at some constant rate γ. The seller bears a per-unit disposal cost $\kappa > 0$. The interpretation of this cost is twofold. Either it is a literal disposal cost, *e.g.*, due to handling or transportation, or it is a per-unit production cost. Under the latter interpretation, the value v^ω of a product of type $\omega \in \{h, l\}$ is the consumer's valuation *net* of production costs.

The products are sent for disposal from the location at 1.[18] This assumption preserves the linear nature of inventory reallocations. The overall structure of the model remains unchanged, but location 1 now makes special "sales" at a negative price of $-\kappa$. We extend the definition of the sorting equilibrium accordingly. As products now also move due to disposal, the inventory quality $\mathbf{q}$ in a sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ must now be sustained by $(\mathbf{p}, \sigma, \gamma)$.

The irrelevance result of Theorem 1 generalizes to this environment. In particular, the seller's problem reduces to selecting the optimal disposal rate γ and the inventory quality q at the outlet threshold. Proposition 3 summarizes the main result for this version of the monopoly problem: the seller clears low-value goods from the stock through only one channel.

Proposition 3. *The seller clears low-value goods from the stock through one channel only. In particular, there exists a threshold disposal cost*

$$\bar{\kappa}(v^h, v^l, \pi) \in \left(\max_{q \in [0,\pi]} \tilde{V}^S(q) - v^l, \frac{\pi}{1-\pi} v^h \right),$$

such that:

(i) if $\kappa < \bar{\kappa}(v^h, v^l, \pi)$, then in any seller-optimal sorting equilibrium, there are no outlets,

(ii) if $\kappa > \bar{\kappa}(v^h, v^l, \pi)$, then in any seller-optimal sorting equilibrium, there is no direct disposal, i.e., $\gamma = 0$.

In addition, $\bar{\kappa}(v^h, v^l, \pi)$ decreases with v^l.

Proof. See Appendix G. □

In words, the seller chooses the cheapest way to clear low-value items from the stock and either delegates this task to consumers, or destroys unsold products herself. Either method is costly: the seller leaves consumer surplus on the table or fails to recover production or disposal costs of her goods. When low-value items become more valuable, direct disposal becomes too expensive. Since markdowns and direct disposal are mutually exclusive, the

[18] For simplicity, location 1 is not available for visiting by consumers.

model immediately implies a positive relationship between transaction prices and disposal rates.

Proposition 3 hence clarifies the types of products for which the model is most applicable. Markdowns arise endogenously to clear unsold inventory for products that: (*i*) have highly uncertain demand, (*ii*) are not easily scalable, so that the seller uses sales information for markdowns but does not adjust production, and (*iii*) have sizable marginal costs, which make direct disposal less attractive than clearance sales.

5 Vintage-Based Pricing

This section introduces a vintage-based markdown process, where an item's price is set deterministically, based on its time in inventory. Despite the change in pricing mechanics, vintage-based pricing yields the same steady-state equilibrium outcomes as the baseline model.

The pricing process in the baseline model is inventory-driven, and best applies to spatial implementations of markdowns. An alternative markdown process, which mirrors Filene's strategy more closely, is based on the current product's vintage. To model this, I assume that the seller has a limited time horizon for selling each unit of her goods. If the product is not sold by reaching this deadline, the seller must dispose of it directly at a per-unit cost $\kappa > 0$. Since I don't have time depreciation in my model, we can normalize this deadline to 1: $X \in [0, 1]$ now represents the vintage of the goods in the seller's stock. Consumers' choice remains the same; they now split their attention between different product vintages.

The key difference between the two models lies in how inventory is distributed: with vintage-based pricing, the distribution of the inventory across X is no longer exogenous. Let $\mu : X \to \mathbb{R}_+$ denote inventory density across vintages. A vintage-based market outcome is a tuple $(\mathbf{p}, \sigma, \mu, \mathbf{q})$ consisting of prices, consumer strategy, inventory distribution, and inventory quality.

We adjust the sorting equilibrium definition accordingly to account for the endogeneity of stock distribution across vintages. A vintage-based sorting equilibrium is a market outcome $(\mathbf{p}, \sigma, \mu, \mathbf{q})$ such that: (*i*) prices and consumer strategy $(\mathbf{p}, \sigma)$ sustain the inventory distribution μ and inventory quality $\mathbf{q}$; and (*ii*) σ maximizes consumer payoff given $(\mathbf{p}, \mathbf{q})$. For formal details, see Section Appendix I.

Theorem 2 establishes the equivalence of the two models.

Theorem 2. $(\mathbf{p}, \mu, \sigma, \mathbf{q})$ *is a vintage-based sorting equilibrium if and only if* $(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ *is a sorting equilibrium for the disposal rate* $\gamma = \mu(1)$.

Proof. See Section Appendix I. □

The equivalence follows from observing that products of any vintage x are either sold to consumers or disposed of. The mass of goods at vintage x therefore coincides with the downstream sales at x in the disposal extension of the model. We obtain that in both models, goods get repriced at the rate of downstream sales. Consequently, we get the exact same predictions for the sustained inventory qualitiess.

6 Extensions and Discussion

This section discusses the limitations of the baseline model and addresses some of them. Section 6.1 incorporates heterogeneous consumers with vertically differentiated preferences for high-value goods: in this model, dynamic pricing helps to screen both consumer types and product types. The other two extensions explore how robust the main results are: Section 6.2 allows for more general consumer strategies, while Section 6.3 considers multiple product values. For brevity, the formal details of these extensions are presented in the Appendices.

6.1 Heterogeneous Consumers

This section extends the baseline model to allow for differentiated consumers. Consumers differ in their willingness to pay for high-value goods, and the seller uses markdowns to both segment consumer types and screen product types. In equilibrium, each consumer type gets a menu of inventory quality and price, resembling Mussa and Rosen (1978).

Consider the baseline model from Section 2, but assume that consumers vary in their willingness to pay for high-value goods. A consumer of type $\theta \in \Theta = [v^l, v^h]$ values a high-value item at θ and a low-quality item at v^l. The distribution of types admits a positive density $f(\cdot)$ over the whole support Θ, with $F(\cdot)$ denoting the cdf.

Each consumer type θ chooses a location to visit and draws a product from there. Let $\mathbf{x} : \Theta \to X$ denote the consumer strategy, mapping each type to a distinct location ($\mathbf{x}$ is injective). The expected payoff of a type-θ consumer in a market outcome $(\mathbf{p}, \mathbf{x}, \mathbf{q})$ is:

$$V^B(\mathbf{p}, \mathbf{x}, \mathbf{q}|\theta) = \mathbf{q}(\mathbf{x}(\theta))(\theta - \mathbf{p}(\mathbf{x}(\theta)))_+ + (1 - \mathbf{q}(\mathbf{x}(\theta)))(v^l - \mathbf{p}(\mathbf{x}(\theta)))_+,$$

In a sorting equilibrium, every consumer type chooses a location optimally: I say that the market outcome $(\mathbf{p}, \mathbf{x}, \mathbf{q})$satisfies *incentive compatibility (IC)* if

$$V^B(\mathbf{p}, \mathbf{x}, \mathbf{q}|\theta) \geq \mathbf{q}(x)(\theta - \mathbf{p}(x))_+ + (1 - \mathbf{q}(x))(v^l - \mathbf{p}(x))_+ \quad \text{(IC)}$$

Proposition 4 shows how the characterization of sorting equilibria generalizes to the model with heterogeneous consumers.

Proposition 4. *For every equilibrium market outcome m with positive sales, there exists a threshold outlet shopper $\hat{\theta} \in (v^l, v^h]$ such that*

(i) consumer types $(\hat{\theta}, v^h]$ shop at non-outlet locations assortatively, i.e. $\mathbf{x}$ *is decreasing on $(\hat{\theta}, v^h]$, and*

(ii) types in $[v^l, \hat{\theta})$ shop at outlet locations.

Moreover, if $\hat{\theta} < v^h$ the payoffs at m depend only on the threshold outlet shopper, i.e. there exist $\nu^B : \Theta \times \Theta \to \mathbb{R}$ and $\nu^{TS} : \Theta \to \mathbb{R}$, such that

$$V^B(m|\theta) = \nu^B\left(\hat{\theta}\middle|\theta\right), \; TS(m) = \nu^{TS}\left(\hat{\theta}\right), \forall \theta \in \Theta, \hat{\theta} \in (v^l, v^h).$$

Proof. See Appendix H. □

Proposition 4 shows how the same sorting mechanism allows the seller to screen both products and consumers. Upstream locations offer better inventory quality and higher prices, and attract higher consumer types.

Moreover, equilibrium payoffs only depend on how consumers split between outlets and non-outlets. I now outline the argument behind this result. Fixing the outlet shopper $\hat{\theta}$, the inventory quality at each location depends only on the mass of consumers visiting upstream locations, as in the baseline model. As consumers sort in descending order, for location $\mathbf{x}(\theta)$ this mass equals $1 - F(\theta)$, which is exogenously given. But then, inventory quality for each consumer type is effectively fixed, which in turn fixes prices through consumer IC constraints.

6.2 Atoms in the Consumer Strategy

The irrelevance result for the baseline model relies on a smooth division of consumer attention across the locations. This section shows that this assumption has bite: once we allow consumers to concentrate attention on finitely many locations, the irrelevance result no longer applies. In particular, for the same level of sorting precision, the seller prefers sorting equilibria with non-atomic consumer strategies, as they allow her to more flexibly reprice unsold inventories as they get sorted through consumer choice.

Let the consumer strategy be described by a cdf $D : [0,1] \to [0,1]$, where $D(x)$ denotes the share of consumers who draw products in $(0, x]$. Unlike the baseline model, D may now have atoms, allowing some locations to attract discrete mass of attention. Proposition 5 shows that the irrelevance result breaks down once we allow for atoms in consumer strategies: the seller cannot achieve her optimal payoff if she strictly prefers active sorting equilibria to neutral sorting.

Proposition 5. *If D admits finitely many discontinuities at non-outlet locations, then the sorting equilibrium* $(\mathbf{p}, D, \mathbf{q}, \gamma)$ *is suboptimal for the seller.*

In words, it is unprofitable for the seller to have a high-priced location with a large customer share. Intuitively, if there is an atom at some non-outlet location, the seller misses on some sorting and repricing opportunities. To illustrate, suppose only two locations are visited: $x = 1/3$ and $x = 2/3$, with the latter an outlet. In this case, sorting fails entirely: the average quality is the same at both locations. This reflects a property of the continuous-time model: consumer purchases are infinitesimal relative to inventory, so that there can be no sorting within a single store at 1/3.[19]

That is, even with a binary product value, the seller prefers to operate infinitely many locations to price-discriminate. This stands in contrast with the classical price discrimination of consumers: there, the seller only needs one menu item per consumer type.

6.3 Multiple Product Types

For tractability, the baseline model assumes binary value for all goods. In this section, I briefly discuss how the model can be generalized beyond binary product values: even a richer environment, the role of prices remains very limited due to tight connection between sales and sorting.

Suppose each product comes in n possible types, yielding consumer values $v^1 < v^2 < \cdots < v^n$. As in the baseline model, consumer strategy admits a density over X. The inventory quality is now defined as $\mathbf{q} : \{1, \ldots, n\} \times [0, 1] \to [0, 1]$, where $\mathbf{q}(i|x)$ is share of value-i goods at location x's stock.

In Online Appendix, I show that the results about the structure of sorting equilibria continue to hold in this richer setting. Proposition 6 generalizes the threshold structure of the sorting equilibrium. Now, each sorting equilibrium features up to n thresholds, where the price schedule crosses one of the products' possible values $\{v^i\}_{i=1}^n$. Proposition 8 in the Online Appendix also formulates a version of the irrelevance result for the model with multiple qualities. Fixing the disposal rate and the set of product types that are ever purchased (at some locations), the set of attainable payoffs remains one-dimensional. The role of prices in the seller's problem remains as limited, as before.

[19] This differs from the two-store numerical example, where time is discrete, so that there is a sizable difference between initial and post-sales quality distribution at the flagship.

Clearly, with richer quality structure, markdowns and direct disposal can coexist: the seller may choose to clear some quality levels through reduced prices, while disposing of the least valuable items directly. To illustrate, suppose there are three quality levels and the lowest has no value to consumers: $v^1 = -\kappa$. Then, sales of quality-1 do not recover any of the production costs. But clearing it through sales requires transferring more surplus to the buyer and reducing transaction prices for all other qualities. Direct disposal of quality-1 items is therefore optimal. At the same time, if v^2 is sufficiently high, then the seller prefers to clear it through sales to recover her production costs.

6.4 Future Directions

In this section, I highlight some of the questions that fall outside the scope of this paper but offer promising avenues for future research.

Quality Depreciation. So far, I have assumed that the preferences for any particular product remain constant over time. But in real life, even popular designs lose customer appeal with time. For instance, in the apparel industry, this may happen due to the seasonality of products. Within this paper, one could accommodate time depreciation by assuming that, with some probability, a unit of unsold high-quality inventory loses its value and becomes of low quality.

With time depreciation, the irrelevance result for the continuous model no longer holds. The seller gets a new leverage for sorting products through their exogenous deterioration and must balance a new trade-off. The seller can "speed up" turnover and dampen the effect of depreciation by increasing the customer share of the earlier locations. By doing so, the seller improves the average inventory quality and increases sales volume at high-priced locations but prevents the goods from getting damaged before reaching outlets.

Other Pricing Processes and Product Flows. My model greatly constrains how the seller leverages information from sales. As argued in the introduction, this assumption may be justified by the high cost of more nuanced pricing strategies for large inventory volumes. Nevertheless, the model leaves the question of optimal pricing and product reallocation open. In particular, would the seller benefit from non-linear reallocation of goods? Could the seller benefit from having two separate, independent lines of stores? Should the seller merge outlets for her different brand lines? Given the tractability model, these directions seem promising within the suggested framework.

Richer Market Structures. Theorem 1 characterizes sorting equilibria for all possible prices, not just the optimal ones. In particular, it applies to richer strategic environments where different sellers manage different prices. In future research, the model could be extended to allow for upstream and downstream sellers to explore whether inefficiency exacerbates with multiple sellers setting prices.

Frequency of Replenishment. The frequency of inventory replenishment offers another strategic tool for the seller to enhance product sorting efficiency. Exploring the impact of replenishment frequency, particularly in scenarios where stock-outs occur, could provide additional valuable insights.

7 Related Literature

This paper builds on the classic model of markdown pricing by Lazear (1986): a seller gradually lowers the price of a good with unknown consumer value, and short-lived consumers arrive gradually. In that setting, the good is produced once, and the seller can extract the entire surplus by waiting long enough. In my model, learning through the lack of sales is costly: unsold goods slow the arrival of new inventory, which is more likely to be high-value. Moreover, with many goods, consumers also make a strategic choice over which goods to inspect. The paper thus offers an equilibrium model of markdowns.

The paper also contributes to the dynamic pricing literature (*e.g.*, Gallego and Van Ryzin (1994), Den Boer (2015), Elmaghraby and Keskinocak (2003), Board and Skrzypacz (2016), Dilme and Li (2019)). In these papers, dynamic prices screen consumers. In mine, they sort products. Methodologically, I differ by focusing on a steady-state equilibrium that reduces the dynamic pricing to a static model.

Prices also serve as signals of expected quality, in the spirit of Wolinsky (1983), Bagwell and Riordan (1991), Delacroix and Shi (2013). However, the source of information is fundamentally different. These papers study informed sellers who use prices to communicate their private information. In contrast, my seller is uninformed, and prices become informative endogenously through the equilibrium sales process.

The sorting mechanism relates to the literature on learning from sales. Bergemann and Välimäki (1997), Bergemann and Välimäki (2000), Bergemann and Välimäki (2006), Bonatti (2011) assume a model of the seller who learns about the product by making sales, and the amount of information increases with the sales volume. Bose et al. (2006), Bose et al. (2008) study dynamic pricing models with information cascades driven by observed purchase history. In contrast, my model makes the absence of purchase a key source of information. This distinction introduces a novel trade-off between sorting precision and sales volume.

The equilibrium model is related to general equilibrium models of directed search with adverse selection (see Guerrieri, Julien, and Wright (2017) for a review). One side of the market has superior information about the match value, and the other chooses the terms of trade. The informed side of the market then sorts across the offered contract. In equilibrium, the terms of contracts get balanced against the probability of matching. In my model, consumers sort themselves and the products across different prices. The resulting trade-off in consumer search lets the seller price-discriminate the goods by quality. Lauermann and Wolinsky (2017) studies how well prices aggregate information in markets with search and adverse selection.

Inventory management under uncertain demand is studied extensively in operations and marketing literature. Some papers focus on how sellers learn and adjust production over time (see Silver, Pyke, and Thomas (2016) for a review). Others explore dynamic demand, where sales influence future outcomes directly (through contagion) or indirectly (through inference) (e.g., Hartung (1973), Petruzzi and Monahan (2003), Caro and Gallien (2007)). These models often treat demand as exogenous and abstract from consumer learning. The most closely related work is an empirical paper by Ngwe (2017), studying the joint pricing and inventory choice problem across a flagship and an outlet for consumer segmentation. Like my model, Ngwe (2017) assumes constant capacity and inventory flow from production to flagship to outlet. However, the model abstracts from consumer search and does not consider how markdowns facilitate indirect quality-based price discrimination.

Methodologically, this paper belongs to the literature on steady-state mechanism design, as in Madsen and Shmaya (2024) and Baccara, Lee, and Yariv (2020).

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Appendices

This section contains the appendices that cover the versions of the model from Section 2 - Section 6. Appendix A formulates the most general version of the model with homogeneous consumers. Appendix B includes all the proofs for this general model. Appendix C specializes to binary product value, and provides proofs for the baseline model, direct disposal model, and a general consumer strategy model. Appendix F provides details and proofs for the reduced seller's problem from Section 4. Appendix H formalizes and analyzes the model with heterogeneous consumers.

The details on the irrelevance result for the model with multiple product values are further deferred to Online Appendix OA2. The proofs for the illustrative two-store example can be found in Online Appendix OA1.

Appendix A General Model

In this section, I formally describe the most general version of the continuous model, which allows for atoms in the consumer strategy, direct disposal, and non-binary product values.

Product Values. The product comes in one of $I \in \mathbf{N}$ possible values. Each consumer gets utility $v^i + \kappa$ from a product of value i, where κ is the per-unit production cost (uniform across value types). For notational simplicity, the values are arranged in ascending order: $v^1 < v^2 < \cdots < v^I$. In addition, it will be useful to define a fictitious value type 0: $v^0 = -\infty$.

Locations and Disposal. Products move across a continuum of locations $X = (0, 1)$ indexing different price tiers, a production plant at 0, and a warehouse at 1. Both 0 and 1 are not available to consumers. The total stock of mass 1 is distributed uniformly across X. The products from location 1 are destroyed at a constant rate $\gamma \geq 0$.

Prices and Inventory Quality. The *inventory quality* is described by $\mathbf{q} : \{1, \ldots, I\} \times [0,1] \to [0,1]$, with $\mathbf{q}(i|x)$ denoting the share of value-i goods in the total stock of location x. The inventory quality at the production plant is given exogenously: $\mathbf{q}(i|0) = \pi(i) > 0$ for some $\{\pi(i)\}_{i=1}^{I}$. $\mathbf{p} : X \to \mathbb{R}$ summarizes the *price schedule* for locations in X, where $\mathbf{p}(x)$ is the seller's price at location x, net of the product's replacement cost $\kappa > 0$. Price schedule $\mathbf{p}(\cdot)$ and every $\mathbf{q}(i|\cdot)$ are Lebesgue-measurable.

Consumers. Consumers who search at location x, draw a single product at random according to distribution $\{\mathbf{q}(i|x)\}_{i=1}^{I}$. If a consumer purchases the product of value type i at a price p, he gets a payoff of $v^i - p$. As before, the consumer buys the good whenever its value is weakly above the price.

The *consumer strategy* is summarized by a cdf $D : [0,1] \to [0,1]$, where $D(x)$ denotes the mass of consumers drawing their good at the locations weakly below x.[20] Define $\delta : X \to [0,1]$ to be the *size of an atom* at location x: $\delta(x) = D(x) - D(x-)$. I assume that consumer strategy admits at most finitely many points of discontinuities. I say that a location x is *visited* if the consumer strategy $D(\cdot)$ is strictly increasing at x.[21]

Payoffs. Analogous to the baseline model, the consumer payoff at a *market outcome* $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ is:

$$V^B(m) \triangleq \int_{x \in X} \sum_{i=1}^{I} \mathbf{q}(i|x)(v^i - \mathbf{p}(x))_+ dD(x)$$

and the total surplus at m is given by the expected value of goods of purchased goods net of the production/diposal cost:

$$TS(m) \triangleq \int_{x \in X} \sum_{i=1}^{I} \mathbf{q}(i|x) v^i \mathbb{1}\{\mathbf{p}(x) \le v^i\} dD(x) - \gamma\kappa.$$

Sustained Inventory Quality. The products move due to purchases, downstream reallocations, and disposal. For every market outcome m, *the purchase probability* $\rho_m : X \to [0,1]$ at a location x is the total probability of drawing a good with the value above the x's price:

$$\rho_m(x) \triangleq \sum_{i=1}^{I} \mathbb{1}\{\mathbf{p}(x) \le v^i\} \mathbf{q}(i|x),$$

[20]Given that D is a cdf over $[0,1]$, I implicitly assume that it is increasing, $D(0) = 0, D(1) = 1$. In addition, D is continuous on the right with a limit on the left (corlol) on $[0,1]$. In addition, since both the location 1 is not available for consumers, D is continuous at 1.

[21]That is, for every $\Delta > 0$: $D(x) - D(x - \Delta) > 0$.

For downstream reallocations, goods pass at the rate of downstream sales $S_m : [0,1] \to [0,1]$, that include fictitious sales from location 1 of size γ:

$$S_m(x) \triangleq \int_{y>x} \rho_m(x) dD(x) + \gamma \mathbb{1}\{x < 1\}.$$

The goods for downstream reallocations are picked at random. On the interval $(x_1, x_2]$, the outflow of products of value i includes all purchases and downstream realoccations of this product type. The inflow of value i equals the mass of value i products reallocated downstream from x_1. The share of value i in the stock of locations $(x_1, x_2]$ stays constant over time when its outflows and inflows are balanced:

$$-\int_{y \in (x_1, x_2], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y) - S_m(x_2)\mathbf{q}(i|x_2) + S_m(x_1)\mathbf{q}(i|x_1) = 0 \tag{3}$$

The inventory quality $\mathbf{q}$ is *sustained* by prices, consumer strategy and disposal rate $(\mathbf{p}, D, \gamma)$ *on a subset* $Y \subseteq [0,1]$ if Equation (3) holds for each value $i \in \{1, \ldots, n\}$ and each $(x_1, x_2] \subseteq Y$. The inventory quality $\mathbf{q}$ is *sustained* by $(\mathbf{p}, D, \gamma)$ if it is sustained on every subset of $[0,1]$.
Sorting Equilibrium. A market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ is *a sorting equilibrium* if

(i) the inventory quality $\mathbf{q}$ is sustained by $(\mathbf{p}, D, \gamma)$,

(ii) each visited location $x \in X$ maximizes consumer expected payoff:

$$\sum_{i=1}^{I} \mathbf{q}(i|x)(v^i - \mathbf{p}(x))_+ = \sup_{y \in X} \sum_{i=1}^{I} \mathbf{q}(i|y)(v^i - \mathbf{p}(y)).$$

In the remainder of the Appendix, I slightly abuse the notation and write $(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ to denote a market outcome where the shopping strategy of a consumer admits a density σ, or write $(\mathbf{p}, D, \mathbf{q})$ to denote a market outcome where $\gamma = 0$.

Appendix B Proofs for the General Model

In this appendix, I analyze the general model from Appendix A. The main result establishes the threshold structure of any sorting equilibrium.

Lemma 4. *For any market outcome m, $S_m(\cdot)$ is corlol. If $D(\cdot)$ is continuous at x, then $S_m(\cdot)$ is also continuous at x.*

Proof. Consider a sequence

$$\Delta_n \to 0^+.$$

For any $x \in [0,1]$ and every n, by the definition of S_m:

$$S_m(x + \Delta_n) - S_m(x) = -\int_{y \in (x, x+\Delta_n] \cap (0,1)} \rho_m(y) dD(y) \tag{4}$$

Since $\rho_m(y) \in [0,1]$ for every $y \in [0,1]$:

$$D(x + \Delta_n) - D(x) \geq \int_{y \in (x, x+\Delta_n] \cap (0,1)} \rho_m(y) dD(y) \geq 0$$

By right continuity of $D(\cdot)$, $D(x+\Delta_n) - D(x) \to 0$, then by Squeeze Theorem, the right-hand-side of Equation (4) converges to 0 as $\Delta_n \to 0$. Then, the left-hand-side also converges to 0, implying right continuity of $S_m(\cdot)$ at x.

Similarly, for every any $x \in [0,1]$ and every n:

$$\begin{aligned} S_m(x - \Delta_n) - S_m(x) &= \int_{y \in (x-\Delta_n, x] \cap (0,1)} \rho_m(y) dD(y) \\ &= \rho_m(x)\delta(x) + \int_{y \in (x-\Delta_n, x) \cap (0,1)} \rho_m(y) dD(y) \end{aligned}$$

Again, since $D(\cdot)$ is left continuous and ρ_m is bounded, $\int_{y \in (x-\Delta_n, x) \cap (0,1)} \rho_m(y) dD(y)$ converges to 0 as $\Delta_n \to 0$, then $S_m(\cdot)$ admits a left limit at x, with:

$$S_m(x-) = S_m(x) + \rho_m(x)\delta(x).$$

If $D(\cdot)$ is continuous at x, then $\delta(x) = 0$, so that $S_m(x-) = S_m(x)$, implying left-continuity of $S_m(\cdot)$ at x. This completes the proof. □

Sustained Inventory Quality. I begin the analysis by characterizing the restrictions on the inventory quality that can be sustained in some sorting equilibrium. Lemma 5 establishes its basic continuity properties.

Lemma 5. *Consider a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$. If $\mathbf{q}$ is sustained by $(\mathbf{p}, D, \gamma)$, then it satisfies the following:*

(i) if $S_m(x) > 0$, then for every i, $\mathbf{q}(i|\cdot)$ is right-continuous at x, and is continuous at x if D is continuous at x,

(ii) if $S_m(x-) > 0$, then for every i, $\mathbf{q}(i|\cdot)$ admits a left limit at x

(iii) if $S_m(x-) > 0$ and either $\mathbf{p}(x) \leq v^1$ or $\mathbf{p}(x) > v^I$, then for every i, $\mathbf{q}(i|\cdot)$ is left-continuous at x

(iv) if $\mathbf{p}(x) > v^i$*, then* $S_m(\cdot)\mathbf{q}(i|\cdot)$ *is continuous at* x*,*

(v) if $S_m(x-) > 0, \mathbf{p}(x) \leq v^i$ *and* D *is discontinuous at* x*, then* $\mathbf{q}(i|x) \leq \mathbf{q}(i|x-)$*, where the inequality is strict whenever* $\mathbf{q}(i|x-) > 0$ *and* $\rho_m(x) < 1$*.*

Proof. Suppose that $\mathbf{q}$ is sustained by $(\mathbf{p}, D, \gamma)$. Then, Equation (3) holds for every $(x_1, x_2] \subseteq (0,1)$ and every value type. Throughout the proof below, I consider a sequence

$$\Delta_n \to 0^+.$$

Part (i). Applying Equation (3) to an interval $(x, x+\Delta_n]$ and a value type i, we must have:

$$-\int_{y\in(x,x+\Delta_n],\mathbf{p}(y)\leq v^i} \mathbf{q}(i|y)dD(y) - S_m(x+\Delta_n)\mathbf{q}(i|x+\Delta_n) + S_m(x)\mathbf{q}(i|x) = 0.$$

As $D(\cdot)$ is right continuous and $\mathbf{q}(i|\cdot)$ is bounded

$$\lim_{n\to\infty} \int_{y\in(x,x+\Delta_n],\mathbf{p}(y)\leq v^i} \mathbf{q}(i|y)dD(y) = 0.$$

Hence, if $\mathbf{q}$ is sustained by $(D, \mathbf{p}, \gamma)$, then we must have:

$$S_m(x)\mathbf{q}(i|x) = \lim_{n\to\infty} S_m(x+\Delta_n)\mathbf{q}(i|x+\Delta_n).$$

By Lemma 4, S_m is right-continuous at every $x \in [0,1]$. Then, to satisfy the above equation, $\mathbf{q}(i|\cdot)$ must also be right-continuous at every location x, where $S_m(x) > 0$.

Analogously, we can show that if $S_m(x) > 0$ and $D(\cdot)$ is left-continuous at x, then $\mathbf{q}(i|\cdot)$ is left-continuous at x.

Part (ii). Applying Equation (3) to an interval $(x-\Delta_n, x]$ for value type i, we must have:

$$\begin{aligned} S_m(x-\Delta_n)\mathbf{q}(i|x-\Delta_n) &= \int_{y\in(x-\Delta_n,x],\mathbf{p}(y)\leq v^i} \mathbf{q}(i|y)dD(y) + S_m(x)\mathbf{q}(i|x) \\ &= \int_{y\in(x-\Delta_n,x),\mathbf{p}(y)\leq v^i} \mathbf{q}(i|y)dD(y) \\ &\quad + (S_m(x) + \delta(x)\mathbb{1}\{\mathbf{p}(x) \leq v^i\})\mathbf{q}(i|x). \end{aligned}$$

As $n \to \infty$, we the right-hand side converges to $(S_m(x) + \delta(x))\mathbf{q}(i|x)$, since $D(\cdot)$ is left-continuous, and $\mathbf{q}(i|y)$ is bounded. Then, the left-hand side also converges, and we obtain:

$$S_m(x-\Delta_n)\mathbf{q}(i|x-\Delta_n) \to (S_m(x) + \delta(x)\mathbb{1}\{\mathbf{p}(x) \leq v^i\})\mathbf{q}(i|x).$$

As $S_m(\cdot)$ admits a left limit by Lemma 4 and as $S_m(x-) > 0$, $\mathbf{q}(i|x-)$ exists and equals:

$$\mathbf{q}(i|x-) = \mathbf{q}(i|x)\frac{(S_m(x) + \delta(x)\mathbb{1}\{\mathbf{p}(x) \le v^i\})}{S_m(x-)}.$$

Part (*iii*). Suppose $\mathbf{p}(x) \le v^1$. Using the proof of part (*ii*), the left limit of $\mathbf{q}(i|\cdot)$ exists at x and equals:

$$\mathbf{q}(i|x-) = \mathbf{q}(i|x)\frac{(S_m(x) + \delta(x)\mathbb{1}\{\mathbf{p}(x) \le v^i\})}{S_m(x-)}.$$

If $\mathbf{p}(x) \le v^1$, then the purchase probability at x equals 1, and

$$S_m(x-) = S_m(x) + \rho_m(x)\delta(x)\mathbb{1}\{\mathbf{p}(x) \le v^i\} = S_m(x) + \delta(x).$$

We then immediately obtain that $\mathbf{q}(i|x-) = \mathbf{q}(i|x)$. The proof for the case $\mathbf{p}(x) > v^I$ is analogous.

Part (*iv*). Suppose that $\mathbf{p}(x) > v^i$: value i is not purchased at x. Then, Equation (3) applied to an interval $(x - \Delta_n, x]$ is equivalent to:

$$S_m(x - \Delta_n)\mathbf{q}(i|x - \Delta_n) - S_m(x)\mathbf{q}(i|x) = \int_{\mathbf{p}(x) \le v^i, y \in (x - \Delta_n, x)} \mathbf{q}(i|y)dD(y).$$

The right-hand side converges to 0. Then, we have:

$$S_m(x - \Delta_n)\mathbf{q}(i|x - \Delta_n) - S_m(x)\mathbf{q}(i|x) \to 0.$$

That is, $S_m(\cdot)\mathbf{q}(i|\cdot)$ is left-continuous at x. Moreover, $S_m(\cdot)$ is right-continuous, and $\mathbf{q}(i|\cdot)$ is right-continuous by part (*i*) whenever $S_m(x) > 0$. Together, these deliver continuity of $S_m(\cdot)\mathbf{q}(i|\cdot)$ at x.

Part (*v*). Suppose that $\mathbf{p}(x) \le v^i$: value i is purchased at x.

From the proof of part (*ii*), we have:

$$\mathbf{q}(i|x) = \mathbf{q}(i|x-)\frac{S_m(x-)}{S_m(x) + \delta(x)\mathbb{1}\{\mathbf{p}(x) \le v^i\}}$$
$$\mathbf{q}(i|x) = \mathbf{q}(i|x-)\frac{\delta(x)\rho_m(x) + S_m(x)}{\delta(x) + S_m(x)}.$$

$\frac{\delta(x)\rho_m(x)+S_m(x)}{S_m(x)+\delta(x)}$ is strictly lower than 1 whenever $\delta(x) > 0, \mathbf{q}(i|x-) > 0$, and $\rho_m(x) < 1$. □

Lemma 6 summarizes the key restrictions on sustained inventory quality. If a value is not purchased over a certain interval, then its mass in downstream reallocations remains constant over this interval. In addition, any two product values are only sorted over a given interval if consumer purchasing decisions are different for these two product types.

Lemma 6. *Consider a market outcome* $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$*. If* $\mathbf{q}$ *is sustained by* $(\mathbf{p}, D, \gamma)$ *on* $[x_1, x_2]$*, then*

(N-i) *if* $\mathbf{p}(x) > v^i$ *D-a.s. on* $[x_1, x_2]$*, then* $S_m(\cdot)\mathbf{q}(i|\cdot)$ *is constant over* $[x_1, x_2]$*.*

(N-ii) *if* $\mathbf{p}(x) \leq v^i$ *D-a.s. on* $[x_1, x_2]$ *and* $S_m(x_2) > 0$*, then* $\mathbf{q}(i|x_1) \leq \mathbf{q}(i|x_2)$*.*

If, additionally, $\mathbf{p}(x) \in (v^i, v^{i+1})$ *D-a.s. on* $[x_1, x_2]$*,* $S_m(x_2) > 0$ *and either D is continuous on* $[x_1, x_2]$*, or* $i = 0$*, then* $\mathbf{q}$ *is sustained by* $(\mathbf{p}, D, \gamma)$ *on* $[x_1, x_2]$ *if and only if:*

(S-i) *for every* $j \leq i$*,* $S_m(\cdot)\mathbf{q}(j|\cdot)$ *is constant over* $[x_1, x_2]$*,*

(S-ii) *for every* $j > i$*,* $\frac{\mathbf{q}(j|\cdot)}{\sum_{k>i}\mathbf{q}(k|\cdot)}$ *is constant over* $[x_1, x_2]$*.*[22]

Proof. **(N-i)**. $\mathbf{q}$ is sustained by $(\mathbf{p}, D, \gamma)$ over $[x_1, x_2]$ if and only if for any $x \in (x_1, x_2]$, and any $\Delta > 0$ such that $(x - \Delta, x] \subseteq [x_1, x_2]$ and $\forall i \in \{1, \ldots, n\}$:

$$\int_{y \in (x-\Delta, x], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y) = S_m(x - \Delta)\mathbf{q}(i|x - \Delta) - S_m(x)\mathbf{q}(i|x). \tag{5}$$

If $\mathbf{p}(\cdot) > v^i$ D-a.s. over $[x_1, x_2]$, then value type i is not purchased over the whole interval, and Equation (5) simplifies to:

$$S_m(x - \Delta)\mathbf{q}(j|x - \Delta) = S_m(x)\mathbf{q}(i|x),$$

for all $x \in (x_1, x_2]$, any all $\Delta > 0$ such that $(x - \Delta, x] \subseteq [x_1, x_2]$. This is equivalent to $S_m(\cdot)\mathbf{q}(i|\cdot)$ being constant over $[x_1, x_2]$.

(N-ii). Since $S_m(x_2) > 0$, by Lemma 5 part (i), $\mathbf{q}(i|\cdot)$ is right-continuous on $[x_1, x_2]$, and by Lemma 5 part (v), if the inventory quality admits a jump, the jump is downwards. Then, if $\mathbf{q}(i|x_2) > \mathbf{q}(i|x_1)$, there exist $y_1, y_2 \in [x_1, x_2]$ such that $\mathbf{q}(i|y_1) = \mathbf{q}(i|x_1)$ and $\mathbf{q}(i|y) > \mathbf{q}(i|x_1), \forall y \in (y_1, y_2]$. If $\mathbf{q}$ is sustained on $[x_1, x_2]$ by $(\mathbf{p}, D, \gamma)$, then from Equation (1) applied to the interval $(y_1, y_2]$:

$$\begin{aligned}\int_{y \in (y_1, y_2]} \mathbf{q}(i|y) dD(y) &= \int_{y \in (y_1, y_2], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y) \\ &= S_m(y_1)\mathbf{q}(i|y_1) - S_m(y_2)\mathbf{q}(i|y_2)\end{aligned}$$

[22] With a convention that $\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)} = 1$ when $\sum_{k>i}\mathbf{q}(k|x) = 0$.

$$= \left(S_m(y_2) + \int_{y\in(y_1,y_2]} \rho_m(y)dD(y)\right)\mathbf{q}(i|y_1) - S_m(y_2)\mathbf{q}(i|y_2)$$

By assumption, $S_m(y_2) \geq S_m(x_2) > 0$, so that we can rewrite the above as follows:

$$\begin{aligned}\mathbf{q}(i|y_2) - \mathbf{q}(i|y_1) &= -\frac{\int_{y\in(y_1,y_2]} (\mathbf{q}(i|y) - \mathbf{q}(i|y_1)\rho_m(y))\, dD(y)}{S_m(y_2)} \\ &\leq -\frac{\int_{y\in(y_1,y_2]} (\mathbf{q}(i|y) - \mathbf{q}(i|y_1))\, dD(y)}{S_m(y_2)} \leq 0,\end{aligned}$$

where the first inequality holds because $\rho_m(\cdot), \mathbf{q}(i|\cdot) \in [0,1]$ and the second inequality holds due to our premise: $\mathbf{q}(i|y) > \mathbf{q}(i|x_1), \forall y \in (y_1, y_2]$. We obtain a contradiction.

Necessity of (S-ii). Fix some product value $j > i$ (purchased D-a.s. on $[x_1, x_2]$) and assume by way of contradiction that there exists a pair of locations $\tilde{x}_1 < \tilde{x}_2$, where $\frac{\mathbf{q}(j|\tilde{x}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_1)} > \frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}$ (the other case is symmetric).

By Lemma 5 (i), $\mathbf{q}(k|\cdot)$ is continuous for every value type k on $[x_1, x_2]$ under the additional premise of the lemma. Then, there exists some $\tilde{y}_1 < \tilde{x}_2$, such that for all $y \in (\tilde{y}_1, \tilde{x}_2]$, $\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} > \frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)}$.[23] $\mathbf{q}$ is sustained by $(D, \mathbf{p}, \gamma)$ over $(\tilde{y}_1, \tilde{x}_2]$ only if:

$$0 = -\int_{y\in(\tilde{y}_1,\tilde{x}_2],\mathbf{p}(y)\leq v^j} \mathbf{q}(j|y)dD(y) + S_m(\tilde{y}_1)\mathbf{q}(j|\tilde{y}_1) - S_m(\tilde{x}_2)\mathbf{q}(j|\tilde{x}_2)$$

By the premise of the lemma, $\mathbf{p}(\cdot) \in (v^i, v^{i+1})$ (D-a.s.) on $[x_1, x_2]$. Then, the purchase probability equals $\sum_{k>i}\mathbf{q}(k|\cdot)$ (D-a.s.) over the whole interval and we can rewrite the above equation as:

$$\begin{aligned}0 = &-\int_{y\in(\tilde{y}_1,\tilde{x}_2],\mathbf{p}(y)\in(v^i,v^{i+1})} \frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)}\rho_m(y)dD(y) + S_m(\tilde{y}_1)\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} \\ &- S_m(\tilde{x}_2)\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)\frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}\end{aligned}$$

By our premise $\frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)} < \frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)}$ for all $y \in (\tilde{y}_1, \tilde{x}_2]$, then we have:

$$\begin{aligned}0 > &-\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)}(S_m(\tilde{y}_1) - S_m(\tilde{x}_2)) + S_m(\tilde{y}_1)\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} \\ &- S_m(\tilde{x}_2)\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)\frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}\end{aligned}$$

[23] For instance, we may take $\tilde{y}_1 = \sup\left\{y : \frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)} = \frac{\mathbf{q}(j|\tilde{x}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_1)}\right\}$.

$$= -\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)}\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)S_m(\tilde{y}_1) + \frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)S_m(\tilde{x}_2)$$
$$+ S_m(\tilde{x}_2)\left(\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} - \frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}\right).$$

From part (N-i), we can substitute $\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)S_m(\tilde{y}_1)$ with $\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)S_m(\tilde{x}_2)$ to get:

$$0 > \left(\frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)} - \frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)}\right)\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)S_m(\tilde{x}_2)$$
$$+ S_m(\tilde{x}_2)\left(\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} - \frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}\right)$$
$$= S_m(\tilde{x}_2)\left(1 - \sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)\right)\left(\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} - \frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}\right).$$

Finally, by our premise, $\frac{\mathbf{q}(j|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} > \frac{\mathbf{q}(j|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)}$. This implies $0 > 0$, a contradiction.

Sufficiency of (S-i) and (S-ii). We now check that Equation (5) is satisfied for every $(x-\Delta, x] \subseteq [x_1, x_2]$ if the conditions in both part (S-i) and (S-ii) are satisfied.

Note that if (S-i) is true, the condition Equation (5) is trivially satisfied for any value type $j \le i$. It remains to show that for every $j > i$, Equation (5) is also satisfied:

$$\int_{y\in(x-\Delta,x],\mathbf{p}(y)\le v^j}\mathbf{q}(j|y)dD(y) - S_m(x-\Delta)\mathbf{q}(j|x-\Delta) + S_m(x)\mathbf{q}(j|x) = 0. \tag{6}$$

Using

$$\rho_m(y) = \sum_{k>i}\mathbf{q}(k|y), \forall y \in [x_1, x_2](D\text{-a.s.})$$

we can rewrite the left-hand-side of Equation (6) as follows:

$$\int_{y\in(x-\Delta,x],\mathbf{p}(y)\le v^j}\mathbf{q}(j|y)dD(y) - S_m(x-\Delta)\mathbf{q}(j|x-\Delta) + S_m(x)\mathbf{q}(j|x)$$
$$= \int_{y\in(x-\Delta,x],\mathbf{p}(y)\in(v^i,v^{i+1})}\frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)}\rho_m(y)dD(y) + S_m(x-\Delta)\mathbf{q}(j|x-\Delta) - S_m(x)\mathbf{q}(j|x) \tag{7}$$

By our premise, the statement of (S-ii) is true: $\frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)}$ stays constant over $[x_1, x_2]$, so that we have:

$$\int_{y\in(x-\Delta,x],\mathbf{p}(y)\in(v^i,v^{i+1})}\frac{\mathbf{q}(j|y)}{\sum_{k>i}\mathbf{q}(k|y)}\rho_m(y)dD(y)=\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)}\int_{y\in(x-\Delta,x],\mathbf{p}(y)\in(v^i,v^{i+1})}\rho_m(y)dD(y)$$
$$=\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)}(S_m(x-\Delta)-S_m(x)),$$

which lets us rewrite Equation (7) as:

$$\int_{y\in(x-\Delta,x],\mathbf{p}(y)\le v^j}\mathbf{q}(j|y)dD(y)-S_m(x-\Delta)\mathbf{q}(j|x-\Delta)+S_m(x)\mathbf{q}(j|x)=$$
$$-\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)}(S_m(x-\Delta)-S_m(x))+S_m(x-\Delta)\sum_{k>i}\mathbf{q}(k|x-\Delta)\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)}$$
$$-S_m(x)\sum_{k>i}\mathbf{q}(k|x)\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)}$$
$$=\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)}\left[S_m(x)\sum_{k\le i}\mathbf{q}(k|x)-S_m(x-\Delta)\sum_{k\le i}\mathbf{q}(k|x-\Delta)\right].$$

If the condition in part (S-i) is satisfied, then $S_m(x)\sum_{k\le i}\mathbf{q}(k|x)-S_m(x-\Delta)\sum_{k\le i}\mathbf{q}(k|x-\Delta)=0$, and we get:

$$\int_{y\in(x-\Delta,x],\mathbf{p}(y)\le v^j}\mathbf{q}(j|y)dD(y)-S_m(x-\Delta)\mathbf{q}(j|x-\Delta)+S_m(x)\mathbf{q}(j|x)=0,$$

as required. □

Sales Collapse. Lemma 7 formalizes the general result: if some value i is never purchased or never destroyed, the sales collapse in the sorting equilibrium. When not cleared from the stock, a value-type crowds out all higher-value items.

Lemma 7. *Consider a market outcome $m=(\mathbf{p},D,\mathbf{q},\gamma)$, such that $\mathbf{p}(x)>v^i$ for all $x<\hat{x}$ (D-a.s.) and $S_m(\hat{x}-)=0$. If m is a sorting equilibrium, then the sales collapse: $S_m(0)=0$.*

Proof. Suppose the statement of the lemma is not true. $S_m(\hat{x}-)=0$, but the total sales are positive: $S_m(0)>0$. Then, by Lemma 6 part (*i*), $S_m(x)\mathbf{q}(i|x)$ is constant on $(0,\hat{x})$ and is right-continuous at 0 (by Lemma 4 and Lemma 5 part (*i*)).

As $\mathbf{q}(i|\cdot)$ is bounded, and $S_m(\hat{x}-)=0$, we must have:

$$S_m(0)\pi(i)=\lim_{x\to\hat{x}}S_m(x)\mathbf{q}(i|x)=0.$$

By assumption, $\pi(i)\in(0,1)$, and we obtain a contradiction with $S_m(0)>0$. □

Imperfect Sorting. Lemma 8 states that the seller can never perfectly discover the value type in a sorting equilibrium with positive sales.

Lemma 8. *Consider a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ with positive sales and with $\mathbf{p}(x) > v^i$ for $x \in (0, \hat{x})$ D-a.s. If $\mathbf{p}(\hat{x}) > v^i$, then $\sum_{k \le i} \mathbf{q}(k|\hat{x}) < 1$ and if $\mathbf{p}(\hat{x}) \le v^i$, then $\sum_{k \le i} \mathbf{q}(k|\hat{x}-) < 1$.*

Proof. Suppose the statement of the lemma is not true. First, assume that there exists some location $x \le \hat{x}$, such that $\mathbf{p}(x) > v^i$, where the discovery of lower values is perfect: $\sum_{k \le i} \mathbf{q}(k|x) = 1$. Let $\tilde{x}$ be the first such location:

$$\tilde{x} \triangleq \inf\{x \le \hat{x} : \sum_{k \le i} \mathbf{q}(k|x) = 1\}.$$

Step 1: $\sum_{k \le i} \mathbf{q}(k|x) = 1$ *for all* $x \in [\tilde{x}, \hat{x})$. By Lemma 6, $\mathbf{q}$ is sustained by $(\mathbf{p}, D, \gamma)$ only if $\sum_{k \le i} \mathbf{q}(k|x) S_m(x)$ remains constant over $[0, \hat{x})$. By the premise of the lemma, total sales are positive, $S_m(0) > 0$. Then, by Lemma 7, we must have $S_m(\hat{x}-) > 0$. As downstream sales are non-increasing, and $\sum_{k \le i} \mathbf{q}(k|x) S_m(x)$ remains constant, then $\sum_{k \le i} \mathbf{q}(k|\cdot)$ is non-decreasing on $[0, \hat{x})$. Then, we must have $\sum_{k \le i} \mathbf{q}(k|x) = 1$ on $(\tilde{x}, \hat{x})$.

In addition, as $S_m(\hat{x}-) > 0$, by Lemma 5 part (i), $\mathbf{q}$ is right-continuous on $[0, \hat{x})$. Then, by the definition of $\tilde{x}$, $\sum_{k \le i} \mathbf{q}(k|\tilde{x}) = 1$.

Step 2: $\sum_{k \le i} \mathbf{q}(k|\cdot)$ *is continuous at* $\tilde{x}$. Suppose not. Then, by Lemma 5 part (i), there is an atom at $\tilde{x}$: $\delta(\tilde{x}) > 0$, and the premise of the lemma is only satisfied if $\mathbf{p}(\tilde{x}) > v^i$.

Since $\sum_{k \le i} \mathbf{q}(k|\tilde{x}) = 1$ and $\mathbf{p}(\tilde{x}) > v^i$, location $\tilde{x}$ makes a zero mass of sales. Then, $S_m(\cdot)$ is continuous at $\tilde{x}$. By Lemma 5 (iii), $S_m(\cdot) \sum_{k \le i} \mathbf{q}(k|\cdot)$ is continuous at $\tilde{x}$, so that $\sum_{k \le i} \mathbf{q}(k|\cdot)$ cannot have a jump at $\tilde{x}$.

Step 3. I now prove the statement on the lemma. By Lemma 6, $\sum_{k \le i} \mathbf{q}(k|x) S_m(x)$ is constant over $[0, \hat{x}]$, and by Step 1, $\sum_{k \le i} \mathbf{q}(k|\tilde{x}) = 1$, so that we obtain:

$$\sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta) S_m(\tilde{x} - \Delta) = \sum_{k \le i} \mathbf{q}(k|\tilde{x}) S_m(\tilde{x}) = S_m(\tilde{x})$$

for any $\tilde{x} > \Delta > 0$. Then, for any such Δ:

$$\frac{1}{\sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta)} = \frac{S_m(\tilde{x} - \Delta)}{S_m(\tilde{x})} = 1 + \frac{\int_{y \in (\tilde{x} - \Delta, \tilde{x}]} \rho_m(y) dD(y)}{S_m(\tilde{x})}.$$

As $\mathbf{p}(\cdot) > v^i$ on $(0, \hat{x})$, then at most value types strictly above i are purchased on $[0, \tilde{x}]$: $\rho_m(y) \le 1 - \sum_{k \le i} \mathbf{q}(k|y)$ for all $y \in (0, \tilde{x}]$. And because $\sum_{k \le i} \mathbf{q}(k|y)$ is non-decreasing (from the proof of Step 1), then $\rho_m(y) \le 1 - \sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)$ for all $y \in [\tilde{x}-\Delta, \tilde{x}]$. Then, we have:

$$\frac{1}{\sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)} \le 1 + (1 - \sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)) \frac{\int_{y\in(\tilde{x}-\Delta,\tilde{x})} \rho_m(y) dD(y)}{S_m(\tilde{x})}.$$

Rearranging, we get:

$$\frac{1 - \sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)}{\sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)} \le \frac{(1 - \sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)) \int_{y\in(\tilde{x}-\Delta,\tilde{x})} \rho_m(y) dD(y)}{S_m(\tilde{x})}.$$

Since $\sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta) < 1$ for every $\Delta > 0$, we can divide both sides by $1 - \sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)$ to obtain:

$$\frac{1}{\sum_{k\le i} \mathbf{q}(k|\tilde{x}-\Delta)} \le \frac{\int_{y\in(\tilde{x}-\Delta,\tilde{x})} \rho_m(y) dD(y)}{S_m(\tilde{x})}.$$

Taking the limit as $\Delta \to 0$, the right-hand side is converging to 0. If the premise is true, the left-hand side must converge to 1 by continuity of $\sum_{k\le i} \mathbf{q}(k|x)$ at $\tilde{x}$ from Step 2. We get a contradiction.

If $\sum_{k\le i} \mathbf{q}(k|x) < 1$ for all $x < \hat{x}$ but converges to 1, the proof is analogous to Step 3.

□

Threshold Structure. Proposition 6 establishes the threshold structure for all sorting equilibria in the general model of Appendix A.

Proposition 6. *Consider a sorting equilibrium* $(\mathbf{p}, D, \mathbf{q}, \gamma)$ *with positive sales. Let* $\hat{x}_i \triangleq \inf\{x \in X : \mathbf{p}(x) \le v^i\}$*, with a convention that* $\hat{x}_i = 1$ *whenever* $\mathbf{p}(x) > v^i$ *for all* $x \in X$*.*

(i) $\hat{x}_i$ *is decreasing in* i*,*

(ii) $\mathbf{p}(\cdot) \in (v^{i-1}, v^i]$ *on* $[\hat{x}_i, \hat{x}_{i-1})$ *(D-a.s.),*

(iii) if the price is strictly above v^1 *at some visited locations, i.e.,* $\int_{y\in(0,\hat{x}_1)} dD(y) > 0$*, then it is weakly above* v^1 *at all visited locations, i.e.,* $\int_{\{y:\mathbf{p}(y)\ge v^1\}} dD(y) = 1$*.*

Proof. **Part** (i). The first part is straightforward, as we take an infimum over a (weakly) larger set as i increases: $\{x \in X : \mathbf{p}(x) \le v^i\} \subseteq \{x \in X : \mathbf{p}(x) \le v^j\}$ for any $j > i$.

Part (ii). For every value type i, define $\tilde{y}_i \in [\hat{x}_i, 1]$ to be the "first" visited location downstream of $\hat{x}_i$ such that the price is strictly above v^i:

$$\tilde{y}_i \triangleq \sup\left\{1 \ge y \ge \hat{x}_i : \int_{z\in(\hat{x}_i,y),\mathbf{p}(z)>v^i} dD(z) = 0\right\}.$$

In what follows, we prove that $\tilde{y}_i = 1$, which is sufficient to confirm part (ii).

Step 1: *Take any location $x \in X$ such that $S_m(x) > 0$ and suppose that there exists a sequence $x_n \to x^+$ such that $\mathbf{p}(x_n) \le v^i$ for every n. Then, the consumer payoff is at least*

$$V^B(m) \ge \sum_{k \ge i} \mathbf{q}(k|x)(v^k - v^i).$$

In a sorting equilibrium, consumer payoff $V^B(m)$ is maximized across all locations, implying:

$$\begin{aligned} V^B(m) &\ge \limsup_{n\to\infty} \sum_{k\ge 1} \mathbf{q}(k|x_n)(v^k - \mathbf{p}(x_n))_+ \\ &\ge \limsup_{n\to\infty} \sum_{k\ge i} \mathbf{q}(k|x_n)(v^k - v^i) = \sum_{k\ge i} \mathbf{q}(k|x)(v^k - v^i), \end{aligned}$$

where the second inequality follows from the $\mathbf{p}(x_n) \le v^i$ by the premise of the statement, and the equality follows from right-continuity of inventory quality at x (due to Lemma 5 (i)).

Step 2: *Consumer payoff at m is at least*

$$V^B(m) \ge \max_{1\le i\le n} \sum_{k\ge i} \mathbf{q}(k|\hat{x}_i)(v^k - v^i) \ge \sum_{k\ge i} \mathbf{q}(k|\hat{x}_1)(v^k - v^1) > 0.$$

First, I verify that $\sum_{k>1} \mathbf{q}(k|\hat{x}_1) > 0$. If $\mathbf{p}(\hat{x}_1) > v^1$, this inequality follows from Lemma 8. If $\mathbf{p}(\hat{x}_1) \le v^1$, then Lemma 8 implies $\sum_{k>1} \mathbf{q}(k|\hat{x}_1-) > 0$. By Lemma 7, if total sales are positive, then $S_m(\hat{x}_1-) > 0$. Then, by Lemma 5 part (iii), inventory quality is left-continuous at $\hat{x}_1$, so that $\sum_{k>1} \mathbf{q}(k|\hat{x}_1) = \sum_{k>1} \mathbf{q}(k|\hat{x}_1-) > 0$.

Next, take an arbitrary i and assume $S_m(\hat{x}_i) > 0$. In this case, $\hat{x}_i$ satisfies the premise of Step 1, so that the consumer payoff is at least $\sum_{k\ge i} \mathbf{q}(k|\hat{x}_i)(v^k - v^i)$.

Now, assume that $S_m(\hat{x}_i) = 0$. Since $\hat{x}_i \le \hat{x}_1$, $S_m(\cdot)$ is non-increasing, then for every i such that $\hat{x}_i < \hat{x}_1$, $S_m(\hat{x}_i) \ge S_m(\hat{x}_1-) > 0$. Then, this case can only arise if $S_m(\hat{x}_1) = 0$ and $\hat{x}_i = \hat{x}_1$, and by Lemma 7, $\mathbf{p}(\hat{x}_i) \le v^1$ (or else the total sales would be zero). Then, the consumer can obtain payoff of at least:

$$V^B(m) \ge \sum_{k\ge 1} \mathbf{q}(k|\hat{x}_1)(v^k - v^1) \ge \sum_{k\ge i} \mathbf{q}(k|\hat{x}_1)(v^k - v^i).$$

Taking stock, we get the desired lower bound on consumer payoff in m:

$$V^B(m) \ge \max_{1\le i\le n} \sum_{k\ge i} \mathbf{q}(k|\hat{x}_i)(v^k - v^i) \ge \sum_{k\ge 1} \mathbf{q}(k|\hat{x}_1)(v^k - v^1) \ge (v^2 - v^1) \sum_{k>1} \mathbf{q}(k|\hat{x}_1) > 0.$$

Step 3(l).: *Take any $y \in X$ such that $\mathbf{p}(y) > v^i$. If there exists a sequence $y_n \to y^-$ such that $\mathbf{p}(y_n) \leq v^i, \forall n$, then $\delta(y) = 0$. If, additionally, $S_m(y) > 0$, then the consumer payoff is at least:*

$$V^B(m) \geq \sum_{k \geq i} \mathbf{q}(k|y)(v^k - v^i).$$

Suppose, by a way of contradiction, that the claim is false. The expected payoff from searching at location y equals:

$$\sum_{k=1}^{I} \mathbf{q}(k|y)(v^k - \mathbf{p}(y))_+.$$

By Step 2, consumer payoff at every visited location is strictly positive, then if y is visited, it has a strictly positive purchase probability. In this case, if $\delta(y) > 0$, then $S_m(y-) > 0$ and by Lemma 5 (v), for any k, such that $v^k - \mathbf{p}(y) \geq 0$, we must have $\mathbf{q}(k|y) \leq \mathbf{q}(k|y-)$. Then, the payoff at y is strictly lower than:

$$\sum_{k=1}^{I} \mathbf{q}(k|y)(v^k - \mathbf{p}(y))_+ < \sum_{k:\mathbf{p}(y) \leq v^k} \mathbf{q}(k|y-)(v^k - v^i) \leq \sum_{k>i} \mathbf{q}(k|y-)(v^k - v^i) \tag{8}$$

However, the consumer can attain a strictly higher payoff by visiting location y_n for some large enough n:

$$V^B(m) \geq \liminf_{n \to \infty} \sum_{n \geq k \geq 1} \mathbf{q}(k|y_n)(v^k - \mathbf{p}(y_n))_+ \tag{9}$$

$$\geq \lim_{n \to \infty} \sum_{k \geq i} \mathbf{q}(k|y_n)(v^k - v^i) = \sum_{k \geq i} \mathbf{q}(k|y-)(v^k - v^i). \tag{10}$$

We obtain a contradiction: y is not visited in m, as it does not deliver the maximal expected payoff.

Since $\delta(y) = 0$, then consumer strategy is continuous at y. If, additionally, $S_m(y) > 0$, then by Lemma 5 (i), inventory quality is continuous at y, and Equation (10) implies $V^B(m) \geq \sum_{k \geq i} \mathbf{q}(k|y)(v^k - v^i)$.

Step 3(r).: *Take any $y \in X$ such that $S_m(y) > 0$ and $\mathbf{p}(y) > v^i$. If there exists a sequence $y_n \to y^+$ such that $\mathbf{p}(y_n) \leq v^i, \forall n$, then $\delta(y) = 0$.*

By Steps 2 and 3, consumer payoff is at least:

$$V^B(m) \geq \max\left\{0, \sum_{k\geq i} \mathbf{q}(k|y)(v^k - v^i)\right\},$$

which is strictly higher than the payoff at y, since $\mathbf{p}(y) > v^i$. Then, y is not visited in a sorting equilibrium m, as it does not deliver a maximal expected payoff to consumers.

Step 4: *either* $\delta(\hat{x}_i) = 0$ *or* $\mathbf{p}(\hat{x}_i) \leq v^i$.

By the proof in Step 2, either $S_m(\hat{x}_i) > 0$, or $\hat{x}_i = \hat{x}_1$ and $\mathbf{p}(\hat{x}_1) \leq v^1$. In the latter case, the claim is trivially true: $\mathbf{p}(\hat{x}_i) = \mathbf{p}(\hat{x}_1) \leq v^1 \leq v^i$. In the former case, $\delta(\hat{x}_i) = 0$ from Step 3 (r) and the definition of $\hat{x}_i$.

Step 5: *If* $\tilde{y}_i < 1$, *then* $S_m(\tilde{y}_i-) > 0$.

By Step 2, consumer payoff is strictly positive, so that consumers do not search at locations with zero purchase probability: $\rho_m(\cdot) > 0$ on $(0, 1)$ D-a.s.. By the definition of $\tilde{y}_i$, if $\tilde{y}_i < 1$, then we must have $\int_{y\in[\tilde{y}_i,1)} dD(y) > 0$, which implies $S_m(\tilde{y}_i-) > 0$.

Step 6: *If* $\tilde{y}_i < 1$, *then either* $\delta(\tilde{y}_i) = 0$, *or* $\mathbf{p}(\tilde{y}_i) \leq v^i$, *and the consumer payoff is at least*

$$V^B(m) \geq \sum_{k\geq i} \mathbf{q}(k|\tilde{y}_i)(v^k - v^i).$$

Case 1: suppose first that $\tilde{y}_i = \hat{x}_i$. Then, the statement in Step 6 is true by Steps 2 and 4.

Case 2: $\tilde{y}_i > \hat{x}_i$. Take any value type k such that $v^k > \mathbf{p}(\tilde{y}_i)$. By the definition of $\tilde{y}_i$ and Step 4, for all $y \in [\hat{x}_i, \tilde{y}_i]$, $\mathbf{p}(y) \leq v^k$ (D-a.s.). Then, by Lemma 6 (N-ii), $\mathbf{q}(k|\hat{x}_i) \geq \mathbf{q}(k|\tilde{y}_i)$, and as $\tilde{y}_i$ is visited, we must have:

$$\begin{aligned} V^B(m) &= \sum_{k:v^k>\mathbf{p}(\tilde{y}_i)} \mathbf{q}(k|\tilde{y}_i)(v^k - \mathbf{p}(\tilde{y}_i)) < \sum_{k:v^k>\mathbf{p}(\tilde{y}_i)} \mathbf{q}(k|\tilde{y}_i)(v^k - v^i) \\ &\leq \sum_{k:v^k>\mathbf{p}(\tilde{y}_i)} \mathbf{q}(k|\hat{x}_i)(v^k - v^i) \\ &\leq \sum_{k\geq i} \mathbf{q}(k|\hat{x}_i)(v^k - v^i), \end{aligned}$$

which contradicts Step 2.

Step 7: *If* $\tilde{y}_i < 1$, *then for any* $\Delta > 0$, $\int_{y\in(\tilde{y}_i,\tilde{y}_i+\Delta],\mathbf{p}(y)>v^i} dD(y) > 0$.

By Step 6, either $\delta(\tilde{y}_i) = 0$, or $\mathbf{p}(\tilde{y}_i) \leq v^i$. But then by the definition of $\tilde{y}_i$, in any right neighborhood of $\tilde{y}_i$, there is a positive mass of visited locations, where the price is strictly above v^i.

Step 8: *If $\tilde{y}_i < 1$, then there exists $\Delta > 0$ such that $\mathbf{p}(\cdot) < v^{i+1}$ over $(\tilde{y}_i, \tilde{y}_i + \Delta]$ D-a.s.*

If the statement is false, there exists a sequence of visited locations $y_n \to \tilde{y}_i^+$ such that $\mathbf{p}(y_n) > v^{i+1}$ for every n. Then, we have

$$V^B(m) = \liminf_{n\to\infty} \sum_{k=1}^{I} \mathbf{q}(k|y_n)(v^k - \mathbf{p}(y_n))_+ \leq \lim_{n\to\infty} \sum_{k\geq i+1} \mathbf{q}(k|y_n)(v^k - v^{i+1})_+$$
$$= \sum_{k\geq i+1} \mathbf{q}(k|\tilde{y}_i)(v^k - v^{i+1}) \leq \sum_{k\geq i+1} \mathbf{q}(k|\hat{x}_i)(v^k - v^{i+1}) < \sum_{k\geq i} \mathbf{q}(k|\hat{x}_i)(v^k - v^i),$$

where the second equality follows from right-continuity of inventory quality at $\tilde{y}_i$, and the second inequality follows from Lemma 6 (*ii*) and $\mathbf{p}(y) < v^{i+1}$ on $[\hat{x}_i, \tilde{y}_i]$ (D-a.s.) (by Steps 4,6 and the definition of $\tilde{y}_i$.). We obtain a contradiction with Step 2.

Step 9: I now show part (*ii*). Suppose that there exists a value-type i, such that $\tilde{y}_i < 1$. Then, by Steps 7 and 8, there exists some $\Delta > 0$, such that $\mathbf{p}(\cdot) < v^{i+1}$ over $(\tilde{y}_i, \tilde{y}_i + \Delta]$ D-a.s. and $\int_{y\in(\tilde{y}_i,\tilde{y}_i+\Delta],\mathbf{p}(y)>v^i} dD(y) > 0$.

Take some visited location $y \in (\tilde{y}_i, \tilde{y}_i + \Delta)$ where $\mathbf{p}(y) > v^i$, then we have:

$$V^B(m) = \sum_{k=1}^{I} \mathbf{q}(k|y)(v^k - \mathbf{p}(y))_+ < \sum_{k\geq i+1} \mathbf{q}(k|y)(v^k - v^i)$$
$$\leq \sum_{k\geq i} \mathbf{q}(k|\hat{x}_i)(v^k - v^i),$$

where the weak inequality follows from $\mathbf{q}(k|y) \leq \mathbf{q}(k|\hat{x}_i)$ for every $k \geq i+1$, due to Lemma 6 (N-ii) and $S_m(y) > 0$. We obtain a contradiction.

Part (*iii*). Suppose there exists some visited location $y \in [\hat{x}_1, 1)$ where $\mathbf{p}(y) < v^1$. By Part (*ii*), locations within the interval $[\hat{x}_1, 1)$ are outlets D-a.s. Then, by Lemma 6 (S-ii) and Lemma 5 (*iii*), inventory quality is constant over all visited locations within $[\hat{x}_1, 1)$: $\mathbf{q}(k|y) = \mathbf{q}(k|\hat{x}_1)$, for every value type k. Then, the consumer payoff is at least:

$$V^B(m) = \sum_{k=1}^{I} \mathbf{q}(k|y)(v^k - \mathbf{p}(y))$$
$$> \sum_{k=1}^{I} \mathbf{q}(k|y)(v^k - v^1) = \sum_{k=1}^{I} \mathbf{q}(k|\hat{x}_1)(v^k - v^1)$$

Let $\tilde{y}$ be the last visited location before $\hat{x}_1$:

$$\tilde{y} = \inf\left\{0 \leq y \leq \hat{x}_1 : \int_{z\in[y,\hat{x}_1)} dD(z) = 0\right\}.$$

Since the measure of locations visited upstream of $\hat{x}_1$ is strictly positive, then $\tilde{y} > 0$. By Lemma 6 (S-ii) and Lemma 5 (*iii*) again, the inventory quality stays constant over $[\tilde{y}, \hat{x}_1]$: $\mathbf{q}(k|\tilde{y}) = \mathbf{q}(k|\hat{x}_1), \forall k$. Then, the location $\tilde{y}$ is not visited, since the price at $\tilde{y}$ is strictly above v^1 but the inventory quality is the same as at $\hat{x}_1$.

In this case, consumer strategy D is continuous at $\tilde{y}$, and by Lemma 5 (*i*), inventory quality is also continuous at $\tilde{y}$. By the definition of $\tilde{y}$, there exists a sequence of visited locations $y_n \to \tilde{y}-$, and by the definition of $\hat{x}_1$, $\mathbf{p}(y_n) \geq v^1, \forall n$. Then, we must have:

$$\begin{aligned} V^B(m) &= \liminf_{n\to\infty} \sum_{k=1}^{I} \mathbf{q}(k|y_n)(v^k - \mathbf{p}(y_n))_+ \leq \lim_{n\to\infty} \sum_{k=1}^{I} \mathbf{q}(k|y_n)(v^k - v^1)_+ \\ &= \sum_{k=1}^{I} \mathbf{q}(k|\tilde{y})(v^k - v^1) = \sum_{k=1}^{I} \mathbf{q}(k|\hat{x}_1)(v^k - v^1)_+, \end{aligned}$$

which delivers a contradiction.

□

Appendix C Binary Product Value

In this section, I provide results for a special case of the model from Appendix A with a binary product value.

Throughout, I refer to following useful observation about the bound on log function.

Observation 1. *For any $q_o \in (0, \pi]$:*

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi) + 1 \leq \frac{\pi}{q_o},$$

where the inequality is strict if $q_o < \pi$.

Proof. As ln is a strictly concave function, then $\ln(y) < y - 1$ for any $y > 1$, so that

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi) + 1 \leq \left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o} - 1\right)(1-\pi) + 1 = \pi\frac{1-q_o}{q_o} + \pi = \frac{\pi}{q_o},$$

where the inequality is strict if $q_o < \pi$. □

Equilibrium Total Surplus. Lemma 9 generalizes Lemma 2 from the main text.

Lemma 9. *Consider a sorting equilibrium $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ with positive total sales, i.e. $S_m(0) > 0$, such that m admits at outlet threshold $\hat{x}$. Then the equilibrium total surplus must* *satisfy the following identity:*

$$TS(m) = S_m(0)\left(\pi v^h + (1-\pi)v^l\right) - \gamma \mathbf{q}(\hat{x})(v^h - v^l) - \gamma(\kappa + v^l).$$

Proof. If $\hat{x}$ is an outlet threshold, then by its definition, either D is continuous at $\hat{x}$ or $\mathbf{p}(\hat{x}) \le v^l$. In either case, $\mathbf{q}$ is continuous at $\hat{x}$ by Lemma 5. By Lemma 6 (S-ii) and Lemma 5 (*iii*), the inventory quality remains constant over $[\hat{x}, 1)$ and coincides with $\mathbf{q}(\hat{x}) = \mathbf{q}(\hat{x}-)$ (D-a.s.). Then, the total surplus is given by:

$$TS(m) = v^h\left(S_m(0) - S_m(\hat{x}-)\right) + \left(\mathbf{q}(\hat{x})v^h + (1-\mathbf{q}(\hat{x}))v^l\right)\left(S_m(\hat{x}-) - \gamma\right) - \gamma\kappa$$

By Lemma 6 (N-i), $S_m(x)(1-\mathbf{q}(x))$ stays constant on $(0,\hat{x})$. Replacing $S_m(\hat{x}-)$ with $S_m(0)\frac{1-\pi}{1-\mathbf{q}(\hat{x})}$, we get:

$$\begin{aligned} TS(m) &= v^h S_m(0)\left(1 - \frac{1-\pi}{1-\mathbf{q}(\hat{x})}\right) + \left[\mathbf{q}(\hat{x})v^h + (1-\mathbf{q}(\hat{x}))v^l\right] S_m(0)\frac{1-\pi}{1-\mathbf{q}(\hat{x})} \\ &- \left[\mathbf{q}(\hat{x})v^h + (1-\mathbf{q}(\hat{x}))v^l\right]\gamma - \gamma\kappa \\ &= S_m(0)(\pi v^h + (1-\pi)v^l) - \gamma\mathbf{q}(\hat{x})(v^h - v^l) - \gamma(\kappa + v^l) \end{aligned}$$

□

Shape of Equilibrium Inventory Quality. For threshold market outcomes, we can characterize the sustained inventory quality using Lemma 6. For what follows, it is useful to provide a definition of the Lambert function $W : \mathbb{R}_{++} \to \mathbb{R}_+$, where $W(x)$ is implicitly defined as:

$$W(x)e^{W(x)} = x.$$

Lemma 10. *Consider some market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ admitting an outlet threshold $\hat{x}$. Letting $D_o \triangleq \int_{x\in[\hat{x},1]} dD(x)$, assume that $D_o + \gamma > 0$ and that the consumer strategy admits discontinuities at $\{x_1, \ldots, x_n\} \subseteq (0,\hat{x})$, and is continuous at $(0,\hat{x}) \setminus \{x_1,\ldots,x_n\}$. Then, the inventory quality $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(\cdot)$ is sustained by $(\mathbf{p}, D, \gamma)$.*

(i) On $(0,\hat{x})$, $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ is recursively defined by:

$$\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x) = \frac{W\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}\exp\left[\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})} - \frac{D(x)-D(x_{i-1})}{(1-q_o)(D_o+\gamma)}\right]\right)}{1 + W\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}\exp\left[\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})} - \frac{D(x)-D(x_{i-1})}{(1-q_o)(D_o+\gamma)}\right]\right)}, \forall x \in [x_{i-1}, x_i)$$

$$\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i) = \left[\left(1 + \frac{(D_o+\gamma)(1-q_o)}{\delta(x_i)(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))}\right) \right.$$
$$\left. - \sqrt{\left(1 + \frac{(D_o+\gamma)(1-q_o)}{\delta(x_i)(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))}\right)^2 - 4\frac{(D_o+\gamma)(1-q_o)\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)}{\delta(x_i)(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))}}\right]/2,$$

taking $x_0 = 0, x_{n+1} = \hat{x}$ *and* q_o*, such that*

$$q_o = \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(\hat{x}-).$$

(ii) On $[\hat{x}, 1]$*, the inventory quality stays constant:*

$$\mathbf{q}(x) = q_o, \forall x \in [\hat{x}, 1].$$

Proof. First, let me verify that $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ is appropriately defined. In particular, we need to show that there exists a unique q_o, such that q: $q_o = \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(\hat{x}-)$. Define two auxiliary functions:

$$\vartheta(q^{(i)}, D, q_o) \triangleq W\left(\frac{q^{(i)}}{1-q^{(i)}} \exp\left[\frac{q^{(i)}}{1-q^{(i)}} - \frac{D}{(1-q_o)(D_o+\gamma)}\right]\right),$$
$$\Psi(\vartheta, q_o) \triangleq \left[\left(1 + \frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)\right) \right.$$
$$\left. - \sqrt{\left(1 + \frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)\right)^2 - 4\frac{(D_o+\gamma)(1-q_o)}{\delta}\vartheta}\right]/2.$$

Letting $q^{(0)}(q_o) = \pi$, define

$$q^{(i+1)}(q_o) \triangleq \Psi\left(\vartheta\left(q^{(i)}(q_o), D(x_{i+1}-) - D(x_i), q_o\right), \delta(x_{i+1}), q_o\right), \forall i \in \{0, \ldots, n-1\}.$$

Then, q_o, by definition, satisfies

$$q_o = \frac{\vartheta(q_n(q_o), D(\hat{x}) - D(x_n), q_o)}{1 + \vartheta(q_n(q_o), D(\hat{x}) - D(x_n), q_o)}. \tag{11}$$

Since both Ψ and ϑ are continuous in q_o, then $\frac{\vartheta(q^{(n)}(q_o), D(\hat{x})-D(x_n), q_o)}{1+\vartheta(q^{(n)}(q_o), D(\hat{x})-D(x_n), q_o)}$ is continuous in q_o as a composition of continuous functions.

I now verify that Ψ is increasing in ϑ and is decreasing in q_o.

$$\partial_\vartheta \Psi = \frac{(D_o+\gamma)(1-q_o)}{2\delta}\left(1 - \frac{\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)-1}{\sqrt{\left(1+\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)\right)^2 - 4\frac{(D_o+\gamma)(1-q_o)}{\delta}\vartheta}}\right) > 0$$

where the inequality follows from

$$\begin{aligned}
&\left(1+\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)\right)^2 - 4\frac{(D_o+\gamma)(1-q_o)}{\delta}\vartheta - \left(\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)-1\right)^2 \\
&= 4\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta) - 4\frac{(D_o+\gamma)(1-q_o)}{\delta}\vartheta > 0.
\end{aligned}$$

Similarly,

$$\partial_{q_o} \Psi = -\frac{D_o+\gamma}{2\delta}\left((1+\vartheta) + \frac{1-\vartheta+(1+\vartheta)\frac{(D_o+\gamma)(1-q_o)}{\delta}}{\sqrt{\left(1+\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)\right)^2 - 4\frac{(D_o+\gamma)(1-q_o)}{\delta}\vartheta}}\right) < 0,$$

where the inequality follows from

$$\begin{aligned}
&(1+\vartheta)^2\left(\left(1+\frac{(D_o+\gamma)(1-q_o)}{\delta}(1+\vartheta)\right)^2 - 4\frac{(D_o+\gamma)(1-q_o)}{\delta}\vartheta\right) \\
&- \left(\vartheta - 1 - (1+\vartheta)\frac{(D_o+\gamma)(1-q_o)}{\delta}\right)^2, \\
&= \left(2+(1+\vartheta)(2+\vartheta)\frac{(D_o+\gamma)(1-q_o)}{\delta}\right)\left(2\vartheta+(1+\vartheta)\vartheta\frac{(D_o+\gamma)(1-q_o)}{\delta}\right) \\
&- 4\vartheta(1+\vartheta)^2\frac{(D_o+\gamma)(1-q_o)}{\delta} > 0.
\end{aligned}$$

In addition, it is easy to see that

$$\begin{aligned}
\partial_{q_o}\vartheta &< 0, \\
\partial_{q^{(i)}}\vartheta &> 0.
\end{aligned}$$

Together, these imply

$$\frac{\vartheta(q^{(n)}(q_o), D(\hat{x}) - D(x_n), q_o)}{1+\vartheta(q^{(n)}(q_o), D(\hat{x}) - D(x_n), q_o)}$$

is decreasing in q_o. As, additionally,

$$0 < \frac{\vartheta(q_n(q_o), D(\hat{x}) - D(x_n), q_o)}{1 + \vartheta(q_n(q_o), D(\hat{x}) - D(x_n), q_o)} \leq \pi,$$

there is a unique $q_o \in [0, \pi]$ satisfying Equation (11).

Second, I verify that $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ is sustained by $(\mathbf{p}, D, \gamma)$ over $[\hat{x}, 1]$. Fix the market outcome $\tilde{m} = (\mathbf{p}, D, \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}, \gamma)$ with the suggested inventory quality. Take any $(x_1, x_2] \subseteq [\hat{x}, 1]$, then we have:

$$\begin{aligned}
&\int_{y\in(x_1,x_2],\mathbf{p}(y)\leq v^h} \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(y)dD(y) - S_{\tilde{m}}(x_2)\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_2) + S_{\tilde{m}}(x_1)\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_1) \\
&= \int_{y\in(x_1,x_2]} q_o dD(y) - S_{\tilde{m}}(x_2)q_o + S_{\tilde{m}}(x_1)q_o \\
&= (S_{\tilde{m}}(x_2) - S_{\tilde{m}}(x_1))q_o - S_{\tilde{m}}(x_2)q_o + S_{\tilde{m}}(x_1)q_o = 0,
\end{aligned}$$

where we used $\mathbf{p}(y) \leq v^h$ on $[\hat{x}, 1]$ (D-a.s.), so that all consumers searching over these locations make a purchase, regardless of the value of the inspected good. Then, $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ is sustained by $(\mathbf{p}, D, \gamma)$ over $[\hat{x}, 1]$ by definition.

Third, I verify that $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ is sustained by $(\mathbf{p}, D, \gamma)$ over $[0, \hat{x}]$. By Lemma 6, it is sufficient to show:

$$(1 - \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x))S_{\tilde{m}}(x) = S_{\tilde{m}}(\hat{x}-)(1 - \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(\hat{x}-)), \forall x \in [0, \hat{x})$$

Given locations $[\hat{x}, 1]$ are outlets D-a.s., $S_{\tilde{m}}(\hat{x}-) = D_o + \gamma$. By its definition, $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(\hat{x}-) = q_o$, so that the above is equivalent to:

$$(1 - \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x))S_{\tilde{m}}(x) = (D_o + \gamma)(1 - q_o), \forall x \in [0, \hat{x}).$$

Step 1: *For every $i \in \{0, \dots, n\}$:*

$$\begin{aligned}
\forall x \in [x_i, x_{i+1}) : S_{\tilde{m}}(x) = &S_{\tilde{m}}(x_{i+1}-) \\
&+ (D_o + \gamma)(1 - q_o)\left(\frac{1}{1 - \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x)} - \frac{1}{1 - \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right).
\end{aligned}$$

As locations $(0, \hat{x})$ are non-outlets (D-a.s.), for any $x \in [x_i, x_{i+1})$:

$$S_{\tilde{m}}(x) = S_{\tilde{m}}(x_{i+1}-) + \int_{y\in(x,x_{i+1})} \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(y)dD(y).$$

For any $x \in [x_n, \hat{x})$:

$$\int_{y\in(x,x_{i+1})} \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(y)dD(y) =$$

$$\int_{y\in(x,x_{i+1})} \frac{W\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \exp\left[\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})} - \frac{D(y)}{(1-q_o)(D_o+\gamma)}\right]\right)}{1+W\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \exp\left[\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})} - \frac{D(y)}{(1-q_o)(D_o+\gamma)}\right]\right)} dD(y)$$

$$= (D_o+\gamma)(1-q_o)\Bigg(W\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \exp\left[\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})} - \frac{D(x)}{(1-q_o)(D_o+\gamma)}\right]\right)$$

$$- W\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \exp\left[\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i-1})} - \frac{D(x_{i+1}-)}{(1-q_o)(D_o+\gamma)}\right]\right)\Bigg)$$

$$= (D_o+\gamma)(1-q_o)\left(\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right).$$

Then, we obtain:

$$S_{\tilde{m}}(x) = S_{\tilde{m}}(x_{i+1}-) + (D_o+\gamma)(1-q_o)\left(\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right)$$

as required.

Step 2: I now prove the statement by induction.

(Initial Case): At $i = n+1$, we have

$$(1-\mathbf{q}(x_{n+1})S_{\tilde{m}}(x_{n+1}-) = (1-q_o)(D_o+\gamma)$$

by construction.

(Induction hypothesis) Suppose that

$$(1-\mathbf{q}(x_{i+1}-)S_{\tilde{m}}(x_{i+1}-) = (1-q_o)(D_o+\gamma).$$

I now show that

$$(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x))S_{\tilde{m}}(x) = (D_o+\gamma)(1-q_o), \forall x \in [x_i, x_{i+1}),$$

and

$$(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))S_{\tilde{m}}(x_i-) = (D_o+\gamma)(1-q_o).$$

For all $x \in [x_i, x_{i+1})$:

$$
\begin{aligned}
(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x))S_{\tilde{m}}(x) &= (1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x))\Bigg[S_{\tilde{m}}(x_{i+1}-) \\
&+ (D_o+\gamma)(1-q_o)\left(\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right)\Bigg] \\
&= (1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x))(D_o+\gamma)(1-q_o)\Bigg[\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)} \\
&+ \left(\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right)\Bigg] \\
&= (D_o+\gamma)(1-q_o),
\end{aligned}
$$

where the first equality is due to Step 1, and the second equality is due to the induction hypothesis.

Finally, I establish the statement regarding $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)$. $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)$ is defined as a solution to the following quadratic equation:

$$
\begin{aligned}
&\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)}\frac{(1-q_o)(D_o+\gamma)}{\delta(x_i)} \\
&= \frac{(1-q_o)(D_o+\gamma)\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{\delta(x_i)(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))} + \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)),
\end{aligned}
$$

which we can equivalently rewrite as:

$$
\begin{aligned}
&\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)(1-q_o)(D_o+\gamma) \\
&= \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)\left((1-q_o)(D_o+\gamma) + (1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i))\delta(x_i)(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))\right).
\end{aligned}
$$

As $(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i))S_{\tilde{m}}(x_i) = D_o+\gamma$, the above implies:

$$
\begin{aligned}
&\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)(1-q_o)(D_o+\gamma) \\
&= (1-q_o)(D_o+\gamma)\left(\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i) + \frac{\delta(x_i)}{S_{\tilde{m}}(x_i)}(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)\right),
\end{aligned}
$$

which simplifies to:

$$
\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-)S_{\tilde{m}}(x_i) = \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)S_{\tilde{m}}(x_i) + (1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)\delta(x_i).
$$

Subtracting $S_{\tilde{m}}(x_i)$ from both sides, we obtain:

$$-(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))S_{\tilde{m}}(x_i) = (1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i))S_{\tilde{m}}(x_i) + \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)\delta(x_i).$$

Rearranging the above, we get:

$$(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))\left(S_{\tilde{m}}(x_i) + \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)\delta(x_i)\right) = (1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_n))S_{\tilde{m}}(x_i).$$

As $S_{\tilde{m}}(x_i-) = S_{\tilde{m}}(x_i) + \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)\delta(x_i)$ and $(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i))S_{\tilde{m}}(x_i) = (D_o+\gamma)(1-q_o)$, then:

$$(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i-))S_{\tilde{m}}(x_i-) = (D_o+\gamma)(1-q_o),$$

as required. This completes the proof: $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ is sustained by $(\mathbf{p}, D, \gamma)$. □

Lemma 11. *Consider a market outcome* $m_1 = (\mathbf{p}, D, \mathbf{q}_1, \gamma)$ *with positive sales admitting an outlet threshold* $\hat{x}$*, such that the inventory quality* $\mathbf{q}$ *is sustained by* $(\mathbf{p}, D, \gamma)$*. If some inventory quality* $\mathbf{q}_1$ *is sustained by* $(\mathbf{p}, D, \gamma)$*, then* $\mathbf{q}_2(x) = \mathbf{q}_1(x)$ *for all* x *such that* $D(x-) < 1$*.*

Proof. Let m_2 be the market outcome with the inventory quality $\mathbf{q}_2$: $m_2 = (\mathbf{p}, D, \mathbf{q}_2, \gamma)$. By Lemma 6 (N-i) and Lemma 5, for both market outcomes we have:

$$S_{m_i}(x)(1-\mathbf{q}_i(x)) = S_{m_i}(\hat{x}-)(1-\mathbf{q}_i(\hat{x})). \tag{12}$$

Since $\hat{x}$ is an outlet threshold, then for both market outcomes, we have:

$$S_{m_i}(\hat{x}-) = D_o \triangleq \int_{y\in[\hat{x},1)} dD(y),$$
$$S_{m_i}(x) = S_{m_i}(\hat{x}-) + \int_{y\in(x,\hat{x})} \mathbf{q}_i(y)dD(y),$$

which lets us rewrite Equation (12) as follows:

$$\left(D_o + \int_{y\in(x,\hat{x})} \mathbf{q}_i(y)dD(y)\right)(1-\mathbf{q}_i(x)) = D_o(1-\mathbf{q}_i(\hat{x})). \tag{13}$$

Let $\Delta_q(x) \triangleq \mathbf{q}_2(x) - \mathbf{q}_1(x)$, then applying the above to the second market outcome, we must have:

$$D_o(1-\mathbf{q}_1(x)) - \Delta_q(\hat{x})D_o = D_o(1-\mathbf{q}_2(\hat{x}))$$

$$
\begin{aligned}
&= \left(D_o + \int_{y\in(x,\hat{x})} \mathbf{q}_2(y)dD(y)\right)(1-\mathbf{q}_2(x)) \\
&= \left(D_o + \int_{y\in(x,\hat{x})} \mathbf{q}_1(y)dD(y)\right)(1-\mathbf{q}_1(x)) \\
&- \Delta_q(x)\left(D_o + \int_{y\in(x,\hat{x})} \mathbf{q}_2(y)dD(y)\right) \\
&+ \left(\int_{y\in(x,\hat{x})} \Delta_q(y)dD(y)\right)(1-\mathbf{q}_2(x)) \\
&= D_o(1-\mathbf{q}_1(x)) \\
&- \Delta_q(x)\left(D_o + \int_{y\in(x,\hat{x})} \mathbf{q}_2(y)dD(y)\right) \\
&+ \left(\int_{y\in(x,\hat{x})} \Delta_q(y)dD(y)\right)(1-\mathbf{q}_2(x)) \\
&= D_o(1-\mathbf{q}_1(x)) \\
&- \Delta_q(x)\frac{D_o(1-\mathbf{q}_2(\hat{x}))}{1-\mathbf{q}_2(x)} \\
&+ \left(\int_{y\in(x,\hat{x})} \Delta_q(y)dD(y)\right)(1-\mathbf{q}_2(x)),
\end{aligned}
$$

where the first equality is from the definition of Δ_q, the second equality from applying Equation (13) to the first market outcome, and the third equality follows from Equation (13) for the second market outcome.

By Lemma 7, $S_m(x) \geq D_o > 0, \forall x \in [0, \hat{x})$, then from the above, we have:

$$
\Delta_q(x) = (1-\mathbf{q}_2(x))\frac{\Delta_q(\hat{x})D_o + \left(\int_{y\in(x,\hat{x})} \Delta_q(y)dD(y)\right)(1-\mathbf{q}_2(x))}{D_o(1-\mathbf{q}_2(\hat{x}))}. \tag{14}
$$

Step 1: suppose that $\Delta(\hat{x}) > 0$.

By Lemma 5 (iii), both inventory qualities are continuous at $\hat{x}$, then $\Delta_q(\cdot)$ is also continuous at $\hat{x}$. Then, if $\Delta_q(\hat{x}) > 0$, the same inequality holds in some left neighborhood of $\hat{x}$.

Take $x_1 < \hat{x}$, such that:

$$
x_1 = \inf\{0 < x : \text{ for all } y \in (x, \hat{x}], \Delta_q(y) > 0\}.
$$

Since both inventory qualities are corlol (by Lemma 5 (i)), then Δ_q is also corlol $\Delta_q(x_1) \geq 0$, and

$$\Delta_q(x_1-) = (1-\mathbf{q}_2(x_1-))\frac{\Delta_q(\hat{x})D_o + \left(\int_{y\in(x_1,\hat{x})}\Delta_q(y)dD(y) + \delta(x_1)\Delta_q(x_1)\right)(1-\mathbf{q}_2(x_1-))}{D_o(1-\mathbf{q}_2(\hat{x}))} > 0,$$

where the inequality follows from $\Delta_q(\hat{x}) > 0$, $\int_{y\in(x_1,\hat{x})}\Delta_q(y)dD(y) > 0$ and $\Delta_q(x_1) \geq 0$.

Then, either $x_1 = 0$, or we get a contradiction with the definition of x_1. If $x_1 = 0$, then

$$\Delta_q(0) = (1-\pi)\frac{\Delta_q(\hat{x})D_o + \left(\int_{y\in(0,\hat{x})}\Delta_q(y)dD(y)\right)(1-\pi)}{D_o(1-\mathbf{q}_2(\hat{x}))} > 0.$$

This contradicts the initial condition on both inventory qualities: $\mathbf{q}_1(0) = \mathbf{q}_2(0) = \pi$, implying $\Delta_q(0) = 0$.

Step 2: suppose that $\Delta_q(\hat{x}) = 0$ and $\Delta_q(x_2) > 0$ for some $x_2 < \hat{x}$. Similarly, let

$$x_3 = \inf\{0 < x < x_2 : \text{ for all } y \in (x, x_1], \Delta_q(y) > 0\}.$$

$$\Delta_q(x_3-) = (1-\mathbf{q}_2(x_3-))\frac{\left(\int_{y\in(x_2,\hat{x})}\Delta_q(y)dD(y) + \int_{y\in[x_2,x_3]}\Delta_q(y)dD(y)\right)(1-\mathbf{q}_2(x_3-))}{D_o(1-\mathbf{q}_2(\hat{x}))} > 0.$$

The inequality follows from $\int_{y\in[x_2,x_3]}\Delta_q(y)dD(y) \geq 0$ (by the definition of x_3) and $\Delta_q(x_2) > 0$ implying $\int_{y\in(x_2,\hat{x})}\Delta_q(y)dD(y) > 0$ (by Equation (14)).

We obtain the same contradiction: $\Delta_q(x_3-) > 0$ implies $x_3 = 0$, but then $\Delta_q(0) > 0$, contradicting the intitial condition on inventory qualities.

The case when either $\Delta_q(\hat{x}) < 0$ or $\Delta_q(x_2) < 0$ for some for some $x_2 < \hat{x}$ is symmetric.

Step 3: $\Delta_q(x) = 0$ for all x such that $D(x-) < 1$. We have shown that $\Delta_q(x) = 0, \forall x \in [0, \hat{x}]$. Then, both inventory qualities coincide at the outlet threshold, and by Lemma 6 (S-ii), $\Delta_q(x) = 0$ for every $x > \hat{x}$, such that $D(x-) < 1$. It only remains to show that if $D(x) = 1$ and $\delta(x) > 0$, then $\Delta_q(x) = 0$. In this case, both inventory qualities are continuous at x by Lemma 5 (*iii*) and the definition of outlet threshold. But then $\Delta_q(x) = 0$, since it is zero in the left neighborhood of x. This completes the proof. □

Bounds of Sorting. Here, I derive the bounds on sorting under a general consumer strategy. Define two boundary functions $\lambda : [0,1] \times [0,1] \to [0,1]$ and $\Lambda : [0,1] \times [0,1] \to [0,1]$. For each $D_o \in [0,1]$ and $\gamma \in [0,1]$, $\lambda(D_o, \gamma)$ is implicitly defined by:[24]

$$1 + \gamma = (D_o + \gamma)(1 - \lambda(D_o, \gamma)) \left[\ln \left(\frac{\pi}{1-\pi} \frac{1 - \lambda(D_o, \gamma)}{\lambda(D_o, \gamma)} \right) + \frac{1}{1-\pi} \right]$$

and

$$\Lambda(D_o, \gamma) \triangleq \pi \frac{D_o + \gamma}{D_o + \gamma + (1 - D_o)(1 - \pi)}.$$

Lemma 12. *Consider market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ with positive total sales admitting an outlet threshold $\hat{x}$. If m is a sorting equilibrium, then*

$$\mathbf{q}(\hat{x}) \in \left[\lambda \left(\int_{y \in [\hat{x}, 1]} dD(y), \gamma \right), \Lambda \left(\int_{y \in [\hat{x}, 1]} dD(y), \gamma \right) \right].$$

In addition,

(i) if consumer strategy admits no atoms at non-outlet locations, i.e. $D(\cdot)$ is continuous on $(0, \hat{x})$, then $\mathbf{q}(\hat{x}) = \lambda \left(\int_{y \in [\hat{x}, 1]} dD(y), \gamma \right)$.

(ii) if there is a unique visited non-outlet location, i.e. there exists $\tilde{x} \in (0, \hat{x})$, such that supp$(D) \cap (0, \hat{x}) = \{\tilde{x}\}$, then $\mathbf{q}(\hat{x}) = \Lambda \left(\int_{y \in [\hat{x}, 1]} dD(y), \gamma \right)$.

(iii) if D is discontinuous on $(0, \hat{x})$ at finitely many points, then $\mathbf{q}(\hat{x}) > \lambda \left(\int_{y \in [\hat{x}, 1]} dD(y), \gamma \right)$.

Proof. Throughout the proof, I fix prices $\mathbf{p}$, consumer strategy D and disposal rate γ. For brevity, let me define:

$$D_o \triangleq \int_{y \in [\hat{x}, 1]} dD(y).$$

As m has total positive sales, then by Lemma 7, $S_m(\hat{x}-) > 0$.

Step 1: only outlets. In this case, all locations are outlets, and by Lemma 6, if $\mathbf{q}$ is sustained by $(\mathbf{p}, D, \gamma)$, then $\mathbf{q}(\hat{x}) = \pi$. Both bounds also collapse to π, as $D_o = 1$: $\Lambda(1, \gamma) = \lambda(1, \gamma) = \pi$. The statement of the lemma is trivially true.

Next, I show the statement of the lemma also holds when non-outlets are visited with positive probability.

[24]Note that $\lambda(D_o, \gamma)$ is appropriately defined, since

$$(D_o + \gamma)(1 - q_o) \left[\ln \left(\frac{\pi}{1-\pi} \frac{1 - q_o}{q_o} \right) + \frac{1}{1-\pi} \right]$$

is continuous and decreasing in q_o, and is in $[D_o + \gamma, \infty)$ for any $q_o \in (0, \pi]$.

Part (ii): Assume there is a unique visited non-outlet location: there exists $\tilde{x} \in (0, \hat{x})$, such that $\text{supp}(D) \cap (0, \hat{x}) = \{\tilde{x}\}$. All locations $[x_1, x_2] \subseteq (0, \tilde{x})$ are outlets D-a.s., and by Lemma 6 (S-ii), inventory quality stays constant over any such interval. By Lemma 5 (i) and (ii), the inventory quality is corlol at $\tilde{x}$, implying:

$$\mathbf{q}(\tilde{x}-) = \mathbf{q}(0) = \pi.$$

As all locations $(\tilde{x}, 1)$ are outlets (D-a.s.), then $S_m(\tilde{x}) = D^o + \gamma$. By the proof of Lemma 5 (ii):

$$\pi = \mathbf{q}(\tilde{x}-) = \mathbf{q}(\tilde{x}) \frac{S_m(\tilde{x}) + \delta(\tilde{x})}{S_m(\tilde{x}-)} = \mathbf{q}(\tilde{x}) \frac{1 + \gamma}{D_o + \gamma + (1 - D_o)\mathbf{q}(\tilde{x})}$$

After rearranging, we obtain:

$$\frac{\pi - \mathbf{q}(\tilde{x})}{(1 - \pi)\mathbf{q}(\tilde{x})} = \frac{1 - D_o}{D_o + \gamma},$$

which simplifies to:

$$\mathbf{q}(\tilde{x}) = \pi \frac{D_o + \gamma}{D_o + \gamma + (1 - D_o)(1 - \pi)}$$

By Lemma 6 and Lemma 5 (i) and (iii), the inventory quality stays constant over the interval $[\tilde{x}, \hat{x}]$:

$$\mathbf{q}(\hat{x}) = \mathbf{q}(\tilde{x}) = \Lambda(D_o, \gamma).$$

Upper bound. For every two non-outlet locations $x_1, x_2 < \hat{x}$, by Lemma 6 (N-i):

$$(1 - \mathbf{q}(x_1))S_m(x_1) = (1 - \mathbf{q}(x_2))S_m(x_2).$$

Locations $(0, \hat{x})$ are non-outlets, hence:

$$S_m(x_1) = S_m(x_2) + \int_{y \in (x_1, x_2]} \mathbf{q}(y) dD(y),$$

and we obtain:

$$(1 - \mathbf{q}(x_1)) \left(S_m(x_2) + \int_{y \in (x_1, x_2]} \mathbf{q}(y) dD(y) \right) = (1 - \mathbf{q}(x_2))S_m(x_2),$$

which simplifies to:

$$\frac{\mathbf{q}(x_1)-\mathbf{q}(x_2)}{(1-\mathbf{q}(x_1))}=\frac{\int_{y\in(x_1,x_2]}\mathbf{q}(y)dD(y)}{S_m(x_2)}\geq\frac{\mathbf{q}(x_2)\left(D(x_2)-D(x_1)\right)}{S_m(x_2)},$$

where the inequality follows from $\mathbf{q}(\cdot)$ being non-increasing on $(0,\hat{x})$ due to Lemma 6 (N-ii).

In particular, taking $x_1=0$ and a limit $x_2\to\hat{x}-$, we obtain:

$$\frac{\pi-\mathbf{q}(\hat{x}-)}{1-\pi}\geq\frac{\mathbf{q}(\hat{x}-)(1-D_o)}{D_o+\gamma}.$$

As $\mathbf{q}(\cdot)$ is continuous at $\hat{x}$ by Lemma 5 (iii), we get from the above:

$$\mathbf{q}(\hat{x})=\mathbf{q}(\hat{x}-)\leq\pi\frac{D_o+\gamma}{D_o+\gamma+(1-\pi)(1-D_o)}=\Lambda(D_o,\gamma).$$

This confirms $\Lambda(D_o,\gamma)$ is the upper bound on $\mathbf{q}(\hat{x})$.

Lower Bound. Let $\{x_1,\ldots,x_n\}\subseteq(0,\hat{x})$ be all non-outlet locations, where the consumer strategy is discontinuous. By Lemma 11, on $[0,\hat{x}]$, the inventory quality coincides with $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$, as defined in Lemma 10.

By construction of $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$, the overall change in the inventory quality over an interval $[x_i,x_{i+1}]$ satisfies:

$$\begin{aligned}&\ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}\frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right)+\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}-\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\\&+\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-))\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}=\frac{D(x_{i+1})-D(x_i)}{(D_o+\gamma)(1-q_o)}\end{aligned}\tag{15}$$

Using the bound from Observation 1:

$$\ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\right)<\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}-1$$

Adding $\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}-\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}$ to both sides of the inequality, we obtain the following bound:

$$\begin{aligned}&\ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}\frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right)+\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}-\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\\&=\frac{D(x_{i+1})-D(x_i)}{(D_o+\gamma)(1-q_o)}-\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-))\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\\&<\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}-1+\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}-\frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\end{aligned}$$

$$= \frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-) - \tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{(1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-))\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})} \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}$$

The inequality above and Equation (15) together imply:

$$\begin{aligned}
&\ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)} \frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\right) + \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})} \\
&< \frac{D(x_{i+1})-D(x_i)}{(D_o+\gamma)(1-q_o)} - \ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}\right) \\
&- \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} + \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1}-)}
\end{aligned}$$

which simplifies to:

$$\ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\right) \tag{16}$$

$$+ \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})} < \frac{D(x_{i+1})-D(x_i)}{(D_o+\gamma)(1-q_o)}. \tag{17}$$

Summing these up across all $i \in \{0, \ldots, n+1\}$, we obtain:

$$\begin{aligned}
&\sum_{i=0}^{n+1} \ln\left(\frac{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} \frac{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}{\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})}\right) \\
&+ \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_i)} - \frac{1}{1-\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}(x_{i+1})} < \sum_{i=0}^{n+1} \frac{D(x_{i+1})-D(x_i)}{(D_o+\gamma)(1-q_o)}
\end{aligned}$$

which simplifies to

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right) + \frac{1}{1-\pi} - \frac{1}{1-q_o} < \frac{1-D_o}{(D_o+\gamma)(1-q_o)}.$$

Part (iii) now follows.

□

Appendix D Convergence: Numerical Simulation

I check if the long-run interpretation of the sustained inventory quality is consistent with the simulations.

Figure 7 plots the evolution of the inventory quality. I discretize time, setting the length of one period (the mass of consumers within a period) to 0.001. $\mathbf{q}_t$ denotes the inventory quality at period t. I assume that at period 0, the inventory quality at all locations is the same and coincides with the production plant π. $\mathbf{q}$ denotes the sustained inventory quality. Simulations depicted in Figure 7 confirm convergence to the steady-state inventory quality $\mathbf{q}$.

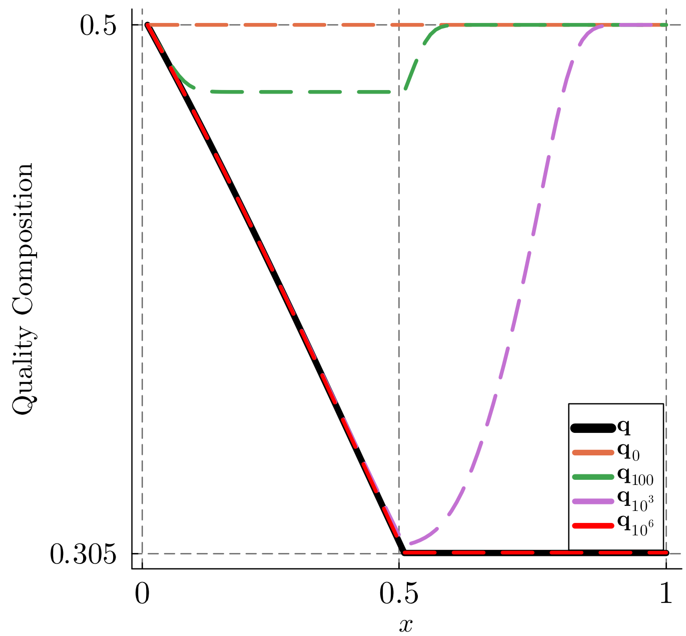

Figure 7: Evolution of the inventory quality

Note: the figure plots the inventory quality after 100, 10^3 and 10^6 periods when 0.5 is the outlet threshold, and the consumer strategy of consumers attention uniformly across all locations at all periods.

Appendix E Omitted Proofs for Section 3

Proof of Lemma 1. Follows from a more general Lemma 6. □

Proof of Theorem 1. The threshold structure of a sorting equilibrium in a binary-quality model follows from a more general Proposition 6.

Part (*i*). Suppose no outlets are visited. In a model with no direct disposal, by Lemma 7, if the measure of visited outlets is zero, then the equilibrium sales are zero. Consequently, both the seller and consumers get a zero equilibrium payoff.

Part (*ii*). Suppose all visited locations are outlets. By Lemma 3 (S-ii), if consumers visit outlets with probability one, then the sorting equilibrium is neutral. Additionally, as (almost) all consumers make purchases, the equilibrium market outcome is efficient. As consumers shop at prices of at most v^l, it follows that the consumer's payoff is at least $\pi(v^h - v^l)$. **Part**

(iii). Consider consumer payoff in a sorting equilibrium, where both types of locations are visited. By Lemma 6 (S-ii), all outlets have the same quality as the outlet threshold $\hat{x}$, D-a.s.. As a positive measure of outlets visited, the consumer's payoff is at least $\mathbf{q}(\hat{x})(v^h - v^l)$. By Proposition 6 part (iii), if non-outlets are visited with positive probability, consumers shop at prices (weakly) above v^l with probability 1. Together, these two imply consumer payoff is exactly $\mathbf{q}(\hat{x})(v^h - v^l)$. Finally, the expression for the total surplus is derived in the main text.

Proof of Lemma 3. If $m = (\mathbf{p}, \sigma, \mathbf{q})$ is a sorting equilibrium with positive sales, then by Lemma 12, $\mathbf{q}(\hat{x}) = \lambda\left(\int_0^1 \sigma(y)dy, 0\right)$, so that:

$$1 = \left(\int_0^1 \sigma(y)dy\right)(1 - \mathbf{q}(\hat{x}))\left[\ln\left(\frac{\mathbf{q}(x)}{1 - \mathbf{q}(x)}\right) + \frac{1}{1-\pi}\right]$$

By Lemma 6 (N-i), since $\mathbf{q}$ is sustained by $(\mathbf{p}, \sigma)$:

$$S_m(0)(1-\pi) = S_m(\hat{x})(1 - \mathbf{q}(\hat{x})) = \left(\int_0^1 \sigma(y)dy\right)(1 - \mathbf{q}(\hat{x})),$$

which implies in a sorting equilibrium m we must have:

$$S_m(0)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1-\pi) + 1\right] = 1,$$

as required. □

Sorting Precision Implementation. To finalize the proof of the theorem, Lemma 13 constructs a sorting equilibrium with any given outlet quality $q \in (0, \pi]$.

Lemma 13. *For any $q \in (0, \pi]$, there exists a sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q})$ with positive total sales that has quality q at the outlet threshold $\hat{x}$: $\mathbf{q}(\hat{x}) = q$.*

Proof. Take any $q \in (0, \pi]$. I construct a sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q})$, such that $\mathbf{q}(\hat{x}) = q$. Take $\sigma(x) = 1, \forall x \in X$. Compute the outlet threshold $\hat{x}$ from:

$$(1-q)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right) = \frac{1}{1-\hat{x}}.$$

Take the inventory quality $\mathbf{q}$ to be $\tilde{\mathbf{q}}_{(\sigma,\hat{x})}$ specified in Lemma 10. Finally, construct prices so as to make consumers indifferent between all locations:

$$\mathbf{p}(x) = v^l, \forall x \in [\hat{x}, 1)$$

$$\mathbf{p}(x) = v^h - \frac{q(v^h - v^l)}{\tilde{\mathbf{q}}_{(\sigma,\hat{x})}(x)}.$$

Then, $\hat{x}$ is indeed the outlet threshold, $\mathbf{q} = \tilde{\mathbf{q}}_{(\sigma,\hat{x})}$ is sustained by $(\mathbf{p}, \sigma)$, and σ is optimal given $(\mathbf{p}, \mathbf{q})$, as required. $\square$

$\square$

Proof of Proposition 1. To prove equivalence between (i) and (ii), use Lemma 12, which implies for both sorting equilibria $m_1 = (\mathbf{p}_1, \sigma_1, \mathbf{q}_1)$ and $m_2 = (\mathbf{p}_2, \sigma_2, \mathbf{q}_2)$:

$$(1 - \mathbf{q}_i(\hat{x})) \left(\int_{\hat{x}}^{1} \sigma_i(x) dx \right) \left[\ln \left(\frac{\pi}{1-\pi} \frac{1 - \mathbf{q}_i(\hat{x})}{\mathbf{q}_i(\hat{x})} \right) + \frac{1}{1-\pi} \right] = 1, \forall i. \tag{18}$$

Then, if the sorting precision in the first equilibrium $\mathbf{q}_1(\hat{x}) < \mathbf{q}_2(\hat{x})$ is higher, then in this equiilibrium more consumers must search at high prices:

$$\int_0^{\hat{x}} \sigma_1(x) dx > \int_0^{\hat{x}} \sigma_2(x) dx.$$

Equivalence between parts (i) and (vi) follows from Theorem 2 (iii).

I now show equivalence between (i) and (v). Note that the average searched non-outlet quality in a sorting equilibrium m_i equals

$$\frac{S_{m_i}(0) - S_{m_i}(\hat{x})}{1 - S_{m_i}(\hat{x}_i)}.$$

By Lemma 1, $(1-\pi)S_{m_i}(0) = (1 - \mathbf{q}_i(\hat{x}_i)) \int_{\hat{x}_i}^1 \sigma_i(x)dx$ and we can rewrite the above as:

$$\frac{S_{m_i}(0) - \frac{1-\pi}{1-\mathbf{q}_1(\hat{x})} S_{m_i}(0)}{1 - \frac{1-\pi}{1-\mathbf{q}_i(\hat{x}_i)} S_{m_i}(0)} = \frac{(\pi - \mathbf{q}_i(\hat{x}_i)) S_{m_i}(0)}{(1 - \mathbf{q}_i(\hat{x}_i)) - (1-\pi) S_{m_i}(0)}.$$

Using identity of Lemma 3, we conclude:

$$\frac{\int_0^{\hat{x}_i} \mathbf{q}_i(x) \sigma_i(x) dx}{\int_0^{\hat{x}_i} \sigma_i(x) dx} = \frac{(\pi - \mathbf{q}_i(\hat{x}_i))}{(1 - \mathbf{q}(\hat{x}_i)) \left[\ln \left(\frac{\pi}{1-\pi} \frac{1-\mathbf{q}(\hat{x}_i)}{\mathbf{q}_i(\hat{x}_i)} \right) (1-\pi) + 1 \right] - (1-\pi)}.$$

Then, to prove the equivalence between (i) and (v), it remains to show that

$$\frac{(\pi - q_o)}{(1 - q_o) \left[\ln \left(\frac{\pi}{1-\pi} \frac{1-q_o}{q_o} \right) (1-\pi) + 1 \right] - (1-\pi)}. \tag{19}$$

is increasing in q_o. Differentiating the above with respect to q_o, one obtains:

$$\partial_{q_o}(19) : -\frac{\left((1-q_o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right]-(1-\pi)\right)}{\left((1-q_o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right]-(1-\pi)\right)^2}$$
$$+\frac{(\pi-q_o)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1+\frac{\pi}{q_o}\right)}{\left((1-q_o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right]-(1-\pi)\right)^2}$$
$$=\frac{-(1-\pi)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right)+(1-\pi)\frac{\pi}{q_o}}{\left((1-q_o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right]-(1-\pi)\right)^2}\geq 0,$$

where the inequality follows from Observation 1.

Finally, I show equivalence between (i) and (iv). The average sales price at non-outlet locations equals the expected total surplus conditional on sales net of conditional consumer surplus. As non-outlets only sell high-quality goods, for any sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q})$ admitting an outlet-threshold $\hat{x}$, the average non-outlet sales price equals:

$$\frac{\int_0^{\hat{x}} \mathbf{p}(x)\mathbf{q}(x)\sigma(x)dx}{\int_0^{\hat{x}} \mathbf{q}(x)\sigma(x)dx} = v_h - \mathbf{q}(\hat{x})(v^h - v^l)\frac{\int_0^{\hat{x}} \sigma(x)dx}{S_m(0) - S_m(\hat{x})}.$$

From the previous step, the above equals:

$$\frac{\int_0^{\hat{x}} \mathbf{p}(x)\mathbf{q}(x)\sigma(x)dx}{\int_0^{\hat{x}} \mathbf{q}(x)\sigma(x)dx} = v^h - \mathbf{q}(\hat{x})(v^h - v^l)\frac{(1-\mathbf{q}(\hat{x}))\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1-\pi)+1\right]-(1-\pi)}{\pi-\mathbf{q}(\hat{x})}.$$

To prove the result, we similarly must show that

$$q_o\frac{(1-q_o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right]-(1-\pi)}{\pi-q_o} \tag{20}$$

is increasing in q_o. Using the previous step the derivative of the above with respect to q_o equals:

$$\partial_{q_o}(20) : \frac{(1-q_o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right]-(1-\pi)}{\pi-q_o}$$
$$+q_o\frac{(1-\pi)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1-\frac{\pi}{q_o}\right)}{(\pi-q_o)^2},$$

which simplifies to:

$$\partial_{q_o}(20) : ((\pi - q_o)(1-q_o) + q_o(1-\pi))\frac{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right)}{(\pi-q_o)^2} - (1-\pi)\frac{2\pi - q_o}{(\pi-q_o)^2}.$$

The sign of $\partial_{q_o}(20)$ hence coincides with the sign of the expression

$$((\pi - q_o)(1-q_o) + q_o(1-\pi))\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right) - (1-\pi)(2\pi - q_o). \quad (21)$$

We need to show that the above is positive for $q_o \in (0, \pi)$. When evaluated $q_o = \pi$, the expression is exactly 0. To complete the proof, I now show that (20) is decreasing in q_o for all $q_o \in (0, \pi)$. Differentiating if with respect to q_o, we obtain:

$$\begin{aligned}
\partial_{q_o}(21) :& -2(\pi - q_o)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right) \\
&-(1-\pi)\frac{(\pi-q_o)(1-q_o)+q_o(1-\pi)}{q_o(1-q_o)} + (1-\pi) \\
=& -2(\pi - q_o)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)(1-\pi)+1\right) + (1-\pi)(\pi-q_o)\frac{2q_o-1}{q_o(1-q_o)} \\
\leq& -2(\pi-q_o) + (1-\pi)(\pi-q_o)\frac{2q_o-1}{q_o(1-q_o)} \\
=&(\pi-q_o)\frac{(2q_o-1)(1-\pi)-2q_o(1-q_o)}{q_o(1-q_o)} \leq -(\pi-q_o)(1-\pi)\frac{1}{q_o(1-q_o)} < 0,
\end{aligned}$$

as required.

□

Appendix F Omitted Proofs for Section 4

In this appendix, I analyze the properties of the seller's payoff as a function of the outlet inventory quality, and characterize the seller's optimum.

Lemma 14. *For any fixed parameter values,* (π, v^h), $\tilde{V}^S(\cdot, \pi, v^h)$ *has the following properties:*

(i) $\tilde{V}^S(\cdot, \pi, v^h)$ *is continuous*

(ii) $\tilde{V}^S(\pi, \pi, v^h) = v^l$

(iii) $\partial_{q-}\tilde{V}^S(\pi, \pi, v^h) > 0$

(iv) $\lim_{q\to 0}\partial_q \tilde{V}^S(q, \pi, v^h) = \infty$

(v) $\tilde{V}^S(\cdot,\pi,v^h)$ *is concave-convex: that is, there exists* $\bar{q}(\pi)\in(0,\pi/2)$*, such that* $\tilde{V}^S$ *is strictly convex on* $(\bar{q}(\pi),\pi]$ *and is strictly concave on* $[0,\bar{q}(\pi))$*.*

(vi) If $\partial_q\tilde{V}^S(\bar{q}(\pi),\pi,v^h)\geq 0$*, then neutral sorting is uniquely optimal*

$$\underset{q\in[0,\pi]}{\mathit{argmax}}\ \tilde{V}^S(q)=\pi.$$

Otherwise, there is a unique interior candidate solution $q_o\in(0,\bar{q}(\pi))$ *satisfying FOC, i.e.* $\partial_q\tilde{V}^s(q_o(\pi,v^h))=0$*, and*

$$\underset{q\in[0,\pi]}{\mathit{argmax}}\ \tilde{V}^S(q)\in\{\pi,q_o(\pi,v^h)\}.$$

Proof. (i) To verify continuity, it is sufficient to check the right limit of $\tilde{V}^S(\cdot,\pi,v^h)$ at 0:

$$\lim_{q\to 0}\tilde{V}^S(q)=\lim_{q\to 0}\frac{\pi v^h+(1-\pi)v^l}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1}=0=\tilde{V}^S(0,\pi,v^h),$$

as required.

(ii) This part of the lemma is straightforward to show by plugging in $q=\pi$ into $\tilde{V}^S(\cdot,\pi,v^h)$.

(iii) The derivative of $V^S(\cdot,\pi,v^h)$ is given by:

$$\partial_q\tilde{V}^S(q,\pi,v^h)=\frac{\pi v^h+(1-\pi)v^l}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}\frac{1-\pi}{(1-q)q}-(v^h-v^l)\tag{22}$$

Plugging in $q=\pi$ into the above, we get:

$$\partial_{q-}\tilde{V}^S(\pi,\pi,v^h)=\frac{\pi v^h+(1-\pi)v^l}{\pi}-(v^h-v^l)=\frac{v^l}{\pi}>0$$

(iv)

$$\lim_{q\to 0}\partial_q\tilde{V}^S(q,\pi,v^h)=\lim_{q\to 0}\frac{\pi v^h+(1-\pi)v^l}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}\frac{1-\pi}{(1-q)q}-(v^h-v^l)$$

I now compute the limit of $\lim_{q\to 0}\frac{1}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}\frac{1}{(1-q)q}$ by applying L'Hôpital's rule twice:

$$\lim_{q\to 0}\frac{1/[(1-q)q]}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}=\lim_{q\to 0}\frac{1}{2(1-\pi)}\frac{(1-2q)/[(1-q)q]}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1}$$
$$=\lim_{q\to 0}\frac{1}{2(1-\pi)^2}\frac{1-2q+2q^2}{(1-q)q}=\infty,$$

as required.

(v) I now show that there exists $\bar{q}(\pi)\in(0,\pi)$, such that $\partial^2_{qq}\tilde{V}^S(q,\pi,v^h)<0$ for all $q\in(0,\bar{q}(\pi))$ and $\partial^2_{qq}\tilde{V}^S(q,\pi,v^h)>0$ for all $q\in(\bar{q}(\pi),\pi)$.

$$\partial^2_{qq}\tilde{V}^S(q,\pi,v^h)=\frac{\pi v^h+(1-\pi)v^l}{(1-q)^2q^2}\,\frac{2(1-\pi)-(1-2q)\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^3}$$

The sign of $\partial^2_{qq}\tilde{V}^S(q,\pi,v^h)$ is then determined by the sign of:

$$2(1-\pi)-(1-2q)\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)\tag{23}$$

When evaluated at any $q\in[\min\{1/2,\pi\},\pi]$, Expression (23) is positive. For $q<1/2$, the Expression (23) is increasing with q, and is negative at $q\to 0$. Then, Expression (23) crosses zero exactly once. Denote the zero of Expression (23) as $\bar{q}(\pi)$.

(vi) By definition, $\partial_q\tilde{V}^S(\cdot,\pi,v^h)$ is minimized at $\bar{q}(\pi)$. Then $\partial_q\tilde{V}^S(\bar{q}(\pi),\pi,v^h)\geq 0$ implies $\tilde{V}^S(\cdot,\pi,v^h)$ is strictly increasing in q on $[0,\pi]$, and neutral sorting is a unique maximum. Else, because $\tilde{V}^S(\cdot,\pi,v^h)$ is strictly concave on $(0,\bar{q}(\pi))$ an by part *(iv)*, $\partial_q\tilde{V}^S(\cdot,\pi,v^h)$ crosses zero exactly once, so that $q_o(\pi,v^h)$ is properly defined for all such parameter values (π,v^h). Finally, since $\tilde{V}^S(\cdot,\pi,v^h)$ it is maximized either at this interior candidate solution, or at the right corner. This completes the proof.

□

Lemma 15. *There exists $\bar{\pi}(v^h)$, such that some neutral sorting equilibrium is seller- optimal if and only if $\pi\leq\bar{\pi}(v^h)$.*

Proof. **Step 1**: There exists some $\bar{\pi}(v^h)$, such that for all $\pi\leq\bar{\pi}(v^h)$, $\tilde{V}^S(\cdot,\pi,v^h)$ is maximized at π.

To prove this claim, I show that $\lim_{\pi\to 0} \partial_q \tilde{V}^S(\bar{q}(\pi), \pi, v^h) = \infty$, and then the statement of Step $i.1$ follows from Lemma 14, part (vi).

$$\partial_q \tilde{V}^S(\bar{q}(\pi), \pi, v^h) = \frac{\pi v^h + (1-\pi)v^l}{4} \frac{(1-2\bar{q}(\pi))^2}{(1-\bar{q}(\pi))\bar{q}(\pi)} - (v^h - v^l),$$
$$\lim_{\pi\to 0} \partial_q \tilde{V}^S(\bar{q}(\pi), \pi, v^h) = \lim_{\bar{q}(\pi)\to 0} \partial_q \tilde{V}^S(\bar{q}(\pi), \pi, v^h) = \infty.$$

Step 2: If $\max_{q\in[0,\pi_1]} \tilde{V}^S(\cdot, \pi_1, v^h) = v^l$, and $\max_{q\in[0,\pi]} \tilde{V}^S(\cdot, \pi, v^h) = v^l$ and $\max_{q\in[0,\pi]} \tilde{V}^S(\cdot, \pi, v^h) > v^l$ in the right neighborhood of π_1, then $\max_{q\in[0,\pi]} \tilde{V}^S(\cdot, \pi, v^h) > v^l$ for any $\pi > \pi_1$.

Suppose by otherwise. As $\tilde{V}^S(\cdot, \pi, v^h)$ is continuous on $[0, \pi] \times (0, \pi) \times [v^l, \infty)$, by Berge's Maximum theorem, $\max_{q\in[0,\pi]} \tilde{V}^S(\cdot, \pi, v^h) = v^l$ is continuous in (π, v^h). Then, if the claim is wrong, there exists some π_2, such that $\max_{q\in[0,\pi]} \tilde{V}^S(\cdot, \pi, v^h) > v^l$ for all $\pi \in (\pi_1, \pi_2)$, with equality on the boundaries. By Lemma 14, parts (ii) and (iv), for $\pi \in (\pi_1, \pi_2)$, the interior candidate is the unique maximum. Then, the interior candidate is properly defined at the boundaries of the interval, too.

Given the definition of the interior candidate solution, By assumption, $\tilde{V}^S(q_o(\pi, v^h), \pi, v^h)$ must be increasing in π at the right neighborhood of π_1 and is decreasing in π in the left neighborhood π_2. By continuity, there exists some $\pi_3 \in (\pi_1, \pi_2)$, such that:

$$\begin{aligned} \frac{d\tilde{V}^S(q_o(\pi_3, v^h), \pi_3, v^h)}{d\pi} &= 0, \\ \frac{d^2\tilde{V}^S(q_o(\pi_3, v^h), \pi_3, v^h)}{d\pi^2} &< 0. \end{aligned} \tag{24}$$

By the definition of the interior solution candidate $q_o(\cdot, \cdot)$:

$$\frac{d\tilde{V}^S(q_o(\pi, v^h), \pi, v^h)}{d\pi} = \partial_\pi \tilde{V}^S(q_o(\pi, v^h), \pi, v^h),$$
$$\frac{d^2\tilde{V}^S(q_o(\pi, v^h), \pi, v^h)}{d\pi^2} = \partial^2_{\pi^2} \tilde{V}^S(q_o(\pi, v^h), \pi, v^h) + \partial^2_{\pi q} \tilde{V}^S(q_o(\pi, v^h), \pi, v^h)\partial_\pi q_o(\pi, v^h).$$

Moreover, by definition, $q_o(\pi, v^h)$ satisfies SOC, so that $\partial^2_{\pi q} \tilde{V}^S(q_o(\pi, v^h), \pi, v^h)\partial_\pi q_o(\pi, v^h) > 0$. Then, the condition of implies:

$$\begin{aligned} \partial_\pi \tilde{V}^S(q_o(\pi_3, v^h), \pi_3, v^h) &= 0, \\ \partial^2_{\pi^2} \tilde{V}^S(q_o(\pi_3, v^h), \pi_3, v^h) &< 0. \end{aligned} \tag{25}$$

However, since we have

$$\partial_\pi \tilde{V}^S(q,\pi,v^h) = \frac{v^h - v^l}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1} + \left(\pi v^h + (1-\pi)v^l\right)\frac{\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) - \frac{1}{\pi}}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}$$
$$= \frac{v^h \ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) - \frac{v^l}{\pi}}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2}.$$

$\partial_\pi \tilde{V}^S(q_o(\pi_3,v^h),\pi_3,v^h) = 0$ implies $\partial^2_{\pi^2}\tilde{V}^S(q_o(\pi_3,v^h),\pi_3,v^h) > 0$, and we get a contradiction.

Step 3. I now complete the proof. By Step 2, if the value function $\max_{q\in[0,\pi_1]} \tilde{V}^S(\cdot,\pi,v^h)$ gets above v^l at some π_1, it must stay above v^l for all $\pi > \pi_1$. By Step 1, neutral sorting is optimal for low enough π. To prove the claim it only remains to show the value function gets above v^l at some $\pi_1 < 1$, so that active sorting is strictly preferred. For sufficiently large π, $\frac{1-\pi}{\pi} \leq \pi$ and hence $\frac{1-\pi}{\pi}$ is a feasible outlet inventory quality:

$$\max_{q\in[0,\pi_1]} \tilde{V}^S(\cdot,\pi,v^h) \geq \tilde{V}^S\left(\frac{1-\pi}{\pi},\pi,v^h\right)$$
$$= \left(\pi v^h + (1-\pi)v^l\right) / \left(2\ln\left(\frac{\pi}{1-\pi}\right)(1-\pi)+1\right) - \frac{1-\pi}{\pi}(v^h - v^l)$$
$$\underset{\pi\to 1}{\to} v^h > v^l,$$

as required. ☐

Lemma 16. *There exists $\bar{v}^h(\pi)$, such that some neutral sorting equilibrium is seller- optimal if and only if $v^h \leq \bar{v}^h(\pi)$.*

Proof. **Step 1**: There exists some $\bar{v}^h(\pi)$, such that for all $v^h < \bar{v}^h(\pi)$, neutral sorting is optimal.

Consider the limit, as the high value approaches the low value:

$$\lim_{v^h\to v^l} \tilde{V}^S(q,\pi,v^h) = \frac{v^l}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)(1-\pi)+1} < v^l, \forall q < \pi.$$

Hence, for sufficiently low v^h, neutral sorting is optimal.

Step 2: $\max_{q\in[0,\pi]} \tilde{V}^S(\cdot,\pi,v^h)$ is increasing in v^h. For all $q \in (0,\pi]$:

$$\partial_{v^h}\tilde{V}^S(q,\pi,v^h) = \frac{\pi}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1} - q$$

By Observation 1, we get a lower bound on the above:

$$\partial_{v^h}\tilde{V}^S(q,\pi,v^h) \geq q - q = 0.$$

It follows that the value function $\max_{q\in[0,\pi]}\tilde{V}^S(\cdot,\pi,v^h)$ is also increasing in v^h.

Step 3. By Berge's Maximum Theorem again, the value function $\max_{q\in[0,\pi]}\tilde{V}^S(\cdot,\pi,v^h)$ is continuous, and by Step 2, if it ever exceeds v^l at some v^h_1, it exceeds v^l for all $v^h > v^1_h$. It remains to verify that for sufficiently high v^h, the value function exceeds v^l.

For sufficiently large v^h, $\frac{v^l}{v^h-v^l}$ is a feasible outlet inventory quality and we have:

$$\max_{q\in[0,\pi]}\tilde{V}^S(\cdot,\pi,v^h) \geq \tilde{V}^S\left(\frac{v^l}{v^h-v^l},\pi,v^h\right) = \frac{\pi v^h+(1-\pi)v^l}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{v^h-2v^l}{v^h-v^l}\right)+1} - v^l$$
$$\underset{v^h\to\infty}{\to} \infty > v^l$$

This completes the proof. □

Proof of Proposition 2. The first part of the proposition follows from Lemma 15 and Lemma 16. I now establish the comparative statics for the the interior solution $q_o(\pi,v^h)$. **Part** (i)**.** Since the interior solution candidate satisfies SOC, the sign of $\partial_{v^h}q_o(\pi,v^h)$ is given by the sign of $\partial^2_{qv^h}\tilde{V}^S(q_o(\pi,v^h),\pi,v^h)$:

$$\partial^2_{qv^h}\tilde{V}^S_{(\pi,v^h)}(q_o(\pi,v^h),\pi,v^h) = \frac{\pi}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o(\pi,v^h)}{q_o(\pi,v^h)}\right)+1\right)^2}\frac{1-\pi}{(1-q_o(\pi,v^h))q_o(\pi,v^h)} - 1$$

Using the definition of q_o, we can replace the first summand in the above to get:

$$\partial^2_{qv^h}\tilde{V}^S_{(\pi,v^h)}(q_o(\pi,v^h),\pi,v^h) = \frac{\pi(v^h-v^l)}{\pi v^h+(1-\pi)v^l} - 1 = -\frac{v^l}{\pi v^h+(1-\pi)v^l} < 0$$

Part (ii). Similarly, the sign of $\partial_\pi q_o(\pi,v^h)$ is determined by the sign of $\partial^2_{q,\pi}\tilde{V}^S_{(\pi,v^h)}(q_o(\pi,v^h))$:

$$\begin{aligned}\partial^2_{q\pi}\tilde{V}^S(q_o(\pi,v^h),\pi,v^h) = {} & \frac{(1-\pi)(v^h-v^l)}{(1-q_o(\pi,v^h))q_o(\pi,v^h)}\frac{1}{\left[\ln\left(\frac{1-\pi}{\pi}\frac{q_o(\pi,v^h)}{1-q_o(\pi,v^h)}\right)(1-\pi)+1\right]^2} \\ & - \frac{\pi v^h+(1-\pi)v^l}{(1-q_o(\pi,v^h))q_o(\pi,v^h)}\frac{1}{\left[\ln\left(\frac{1-\pi}{\pi}\frac{q_o(\pi,v^h)}{1-q_o(\pi,v^h)}\right)(1-\pi)+1\right]^2}\end{aligned}$$

$$+ 2\frac{\pi v^h + (1-\pi)v^l}{(1-q_o(\pi, v^h))q_o(\pi, v^h)} \frac{(1-\pi)\ln\left(\frac{1-\pi}{\pi}\frac{q_o(\pi,v^h)}{1-q_o(\pi,v^h)}\right) - \frac{1-\pi}{\pi}}{\left[\ln\left(\frac{1-\pi}{\pi}\frac{q_o(\pi,v^h)}{1-q_o(\pi,v^h)}\right)(1-\pi)+1\right]^3}.$$

After simplifying the above, we obtain that the sign of the above coincides with the sign of

$$\begin{aligned}
&\pi v^h - 2\frac{\pi v^h + (1-\pi)v^l}{\left[\ln\left(\frac{1-\pi}{\pi}\frac{q_o(\pi,v^h)}{1-q_o(\pi,v^h)}\right)(1-\pi)+1\right]} \\
=&\pi v^h + q_o(\pi, v^h)(v^h - v^l) - 2\tilde{V}^S(q_o(\pi, v^h), \pi, v^h).
\end{aligned} \tag{26}$$

$q_o(\pi, v^h) \leq \pi \leq 1$ and using the proof of Lemma 15:

$$\lim_{\pi \to 1} \tilde{V}^S(q_o(\pi, v^h), \pi, v^h) = v^h,$$

so that for large enough π, Expression (26) gets negative, as required.

□

Appendix G Omitted proofs for Section 4.1

Proof of Proposition 3. By Proposition 6, a sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ is $\hat{x}$-threshold market outcome. For brevity, let

$$q_o \triangleq \mathbf{q}(\hat{x}),$$
$$D_o \triangleq \int_{\hat{x}}^1 \sigma(y)dy.$$

By Lemma 9, the total surplus in m is given by:

$$TS(m) = S_m(0)(\pi v^h + (1-\pi)v^l) - \gamma q_o(v^h - v^l) - \gamma(\kappa + v^l).$$

By Lemma 6 and Lemma 5: $S_m(0) = \frac{S_m(\hat{x})(1-q_o)}{1-\pi} = (D_o + \gamma)(1 - q_o)$. By Lemma 12, as the consumer strategy is continuous in $(0, \hat{x})$, then $q_o = \lambda(D_o, \gamma)$. Combining all these, we obtain:

$$TS(m) = (1+\gamma)\frac{\pi v^h + (1-\pi)v^l}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right)+1} - \gamma q_o(v^h - v^l) - \gamma(\kappa + v^l).$$

The seller's payoff is the difference between the total surplus and the consumer payoff:

$$V^S(\mathbf{p}, \sigma, \mathbf{q}, \gamma) = TS(\mathbf{p}, \sigma, \mathbf{q}, \gamma) - V^B(\mathbf{p}, \sigma, \mathbf{q}, \gamma).$$

Similar to Theorem 1, we now consider three cases of the sorting equilibria, depending on the consumer share of outlets.

Case 1: *only non-outlets are visited.* In this case, $\hat{x} = 1$, $D_o = 0$, and the seller may extract the whole total surplus from a market outcome by charging a price of v^h at all store locations.

By Lemma 12, the inventory quality at 1 is given by $\lambda(0, \gamma)$, and we can rewrite the total surplus at m again as:

$$\begin{aligned} TS(m) &= (\pi v^h + (1-\pi)v^l)\gamma\frac{\lambda(0,\gamma)}{1-\pi} - \gamma(1-\lambda(0,\gamma))(v^h - v^l) - \gamma(\kappa + v^l) \\ &= \frac{\pi v^h}{1-\pi}\gamma\lambda(0,\gamma) - \gamma(1-\lambda(0,\gamma))v^h - \gamma\kappa. \end{aligned}$$

Hence, we can write the seller's profit for a sorting equilibrium with no outlets as:

$$\hat{V}^S_\kappa(\gamma) = \frac{\pi v^h}{1-\pi}\gamma\lambda(0,\gamma) - \gamma(1-\lambda(0,\gamma))v^h - \gamma\kappa.$$

Denote the seller's maximal profit from such sorting equilibria as V^{**}:

$$V^{**} = \sup_{\gamma > 0} \hat{V}^S(\gamma, \kappa).$$

Case 2: *only outlets are visited.* Whenever consumers shop only at outlet locations, the seller receives a constant price $\bar{p} \leq v^l$, and the seller's profit is at most v^l, which is equal to $\tilde{V}^S(\pi)$.

Case 3: *both types of locations are visited.* If consumers shop at both outlets and non-outlets, then by Proposition 6 (*iii*), consumer payoff is exactly $V^B(m) = q_o(v^h - v^l)$ and we get:

$$\begin{aligned} V^S(m) &= TS(m) - V^B(m) \\ &= \left(\pi v^h + (1-\pi)v^l\right)\frac{1+\gamma}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q_o}{q_o}\right) + 1} \\ &\quad - \gamma q_o(v^h - v^l) - \gamma(\kappa + v^l) - q_o(v^h - v^l) \\ &= (1+\gamma)\tilde{V}^S(q_o) - \gamma(\kappa + v^l) \end{aligned}$$

Hence, the seller's maximal profit among all sorting equilibria where consumers shop at both types of locations is:

$$V^* = \sup_{\gamma \geq 0} \sup_{\substack{q \in [\lambda(0,\gamma),\pi] \\ q>0}} (1+\gamma)\tilde{V}^S(q) - \gamma(\kappa + v^l)$$

Clearly, $V^* \geq v^l = \tilde{V}^S(\pi)$, which the seller can achieve with no direct disposal by setting $\gamma = 0$. Then, the seller's optimal choice reduces to selecting between V^* and V^{**}.

Let q^* be the optimizer of $\tilde{V}^S$ over $(0,\pi]$. I now establish the following:

$$\max\{V^*, V^{**}\} = \begin{cases} \max\limits_{\gamma \in (0,\infty)} \hat{V}^S_\kappa(\gamma), \text{ if } \tilde{V}^S(q^*) > \kappa + v^l \\ \tilde{V}^S(q^*), \text{ if } \pi/(1-\pi)v^h \leq \kappa \\ \max\left\{\tilde{V}^S(q^*), \max\limits_{\gamma \in (0,\infty)} \hat{V}^S_\kappa(\gamma)\right\}, \text{ if } \kappa \in \left[\tilde{V}^S(q^*) - v^l, \frac{\pi}{1-\pi}v^h\right) \end{cases}$$

Case 1: $\tilde{V}^S(q^*) > \kappa + v^l$. Then, the seller is better off not having outlet locations: $V^{**} > V^*$.

Note first that $\lambda(0,\gamma)$ is increasing γ and:

$$\lim_{\gamma \to 0} \lambda(0,\gamma) = 0, \ \lim_{\gamma \to \infty} \lambda(0,\gamma) = \pi.$$

If $\tilde{V}^S(q^*) > \kappa + v^l$, then, with no disposal, the seller's profit is maximized at an active sorting equilibrium: $q^* < \pi$. There exists a unique $\tilde{\gamma} \in (0,\infty)$, such that the seller achieves exactly the sorting precision that maximizes $\tilde{V}^S(\cdot)$:

$$\lambda(0,\tilde{\gamma}) = q^*.$$

For all lower disposal rates $\gamma \leq \tilde{\gamma}$ and all outlet qualities $q \in (0,\pi]$:

$$\begin{aligned} (1+\gamma)\tilde{V}^S(q) - \gamma(\kappa + v^l) &\leq (1+\gamma)\tilde{V}^S(q^*) - \gamma(\kappa + v^l) \\ &\leq (1+\tilde{\gamma})\tilde{V}^S(q^*) - \tilde{\gamma}(\kappa + v^l) \\ &= (1+\tilde{\gamma})\tilde{V}^S(\lambda(0,\tilde{\gamma})) - \tilde{\gamma}(\kappa + v^l) \\ &< \left(\frac{\pi}{1-\pi}v^h\tilde{\gamma}\lambda(0,\tilde{\gamma}) - (1-\lambda(0,\tilde{\gamma}))v_h\tilde{\gamma} - \tilde{\gamma}\kappa\right) \leq V^{**}. \end{aligned}$$

Similarly, for all higher disposal rates $\gamma > \tilde{\gamma}$:

$$\sup_{\substack{q \in [\lambda(0,\gamma),\pi] \\ q>0}} (1+\gamma)\tilde{V}^S(q) - \gamma(\kappa + v^l) = (1+\gamma)\max\{\tilde{V}^S(\pi), \tilde{V}^S(\lambda(0,\gamma))\} - \gamma(\kappa + v^l) < V^{**}.$$

where I use the fact that $\tilde{V}^S$ is convex-concave by Lemma 14, and hence whenever the lower bound is binding, $\tilde{V}^S$ reaches its optimum at one of the corners. That is, $\tilde{V}^S(q^*) > \kappa + v^l$ is sufficient for the seller not to use outlet locations.

Case 2. Alternatively, suppose $\tilde{V}^S(q^*) \leq \kappa + v^l$, then:

$$\sup_{\substack{q\in[\lambda(0,\gamma),\pi] \\ q>0}} (1+\gamma)\tilde{V}^S(q) - \gamma(\kappa + v^l) \leq (1+\gamma)\tilde{V}^S(q^*) - \gamma(\kappa+v^l) \leq \tilde{V}^S(q^*)$$

That is, in this case, $V^* = \tilde{V}^S(q^*)$. The seller does not use direct disposal simultaneously with outlet locations. Additionally, for $\pi/(1-\pi)v^h \leq \kappa$, is a sufficient condition for the optimality of outlets, as in this case

$$V^* \geq \tilde{V}^S(\pi) = v^l > 0 = V^{**}.$$

Optimal Disposal Rate. Let me now verify that an optimal $\hat{V}^S$ attains its optimum on $(0,\infty)$ for any $\kappa > 0$ whenever $\pi/(1-\pi)v^h > \kappa$. To that end, we examine the limit $\partial_\gamma \hat{V}^S_\kappa(\gamma)$ at the two corners.

$$\partial_\gamma \hat{V}^S_\kappa(\gamma) = \frac{\pi}{1-\pi}v^h(1-\lambda(0,\gamma)) - \lambda(0,\gamma)v_h - \kappa - \frac{\gamma v^h}{1-\pi}\partial_\gamma\lambda(0,\gamma)$$

An unbounded disposal rate is suboptimal:

$$\partial_\gamma \hat{V}^S_\kappa(\gamma) \underset{\gamma\to\infty}{\to} -\kappa - \lim_{\gamma\to\infty}\frac{\gamma v^h}{1-\pi}\partial_\gamma\lambda(0,\gamma) \leq -\kappa$$

as $\partial_\gamma\lambda(0,\gamma) > 0$.

Consider now the other bound:

$$\partial_\gamma \hat{V}^S_\kappa(\gamma) \underset{\gamma\to 0}{\to} \frac{\pi}{1-\pi}v^h - \kappa - \lim_{\gamma\to 0}\frac{\gamma v^h}{1-\pi}\partial_\gamma\lambda(0,\gamma)$$

Then, to establish the optimal choice of γ is strictly above 0 for any $\pi/(1-\pi)v^h > \kappa$, it is enough to show $\lim_{\gamma\to 0}\gamma\partial_\gamma\lambda(0,\gamma) = 0$.

$$\gamma\partial_\gamma\lambda(0,\gamma) = \frac{1}{(1+\gamma)/(1-\lambda(0,\gamma)) + \gamma/\lambda(0,\gamma)}$$

Hence, the limit $\lim_{\gamma\to 0}\gamma\partial_\gamma\lambda(0,\gamma) = 0$ is determined by the limit of $\gamma/\lambda(0,\gamma)$. From the definition of Φ, it must be that γ converges to 0 at the same rate as $\ln(\lambda(0,\gamma))$, hence $\lim_{\gamma\to 0}\frac{\gamma}{\lambda(0,\gamma)} = \infty$ implying $\lim_{\gamma\to 0}\gamma\partial_\gamma\lambda(0,\gamma) = 0$ as required.

Regime Switches. Finally, to establish there is a unique threshold where the optimal regime switches, note that $\tilde{V}^S(q^*)$ is independent of κ whereas $\max_{\gamma\in(0,\infty)} \hat{V}^S_\kappa(\gamma)$ is strictly decreasing in κ. Hence, for every parameters (v^h, v^l, π), there exists $\bar{\kappa}$ as in the formulation of the proposition.

To prove $\bar{\kappa}$ is increasing in v^l, note that $\max_{\gamma\in(0,\infty)} \hat{V}^S_\kappa(\gamma)$ is constant in v^l, but $\tilde{V}^S(q^*)$ is strictly increasing in v^l. Hence, if v^l increases, the switch occurs at a lower production/disposal cost.

□

Appendix H Omitted Proofs for Section 6.1

In the model with heterogeneous consumers, we may describe the product flows by letting:

$$D_m(x) = \int_{\mathbf{x}(\theta)\leq x:\mathbf{p}(\mathbf{x}(\theta))\leq\theta} dF(\theta)$$

denote the mass of consumers drawing goods from locations $[0, x]$ in a market outcome $(\mathbf{p}, \mathbf{x}, \mathbf{q})$. Then, $\mathbf{q}$ is sustained by $(\mathbf{p}, \mathbf{x})$ in the heterogeneous consumers extensions if and only if it is sustained in the baseline model by $(\mathbf{p}, D_m)$.

A market outcome $m = (\mathbf{p}, \mathbf{x}, \mathbf{q})$ is *a sorting equilibrium* if (i) $\mathbf{q}$ is sustained by $(\mathbf{p}, \mathbf{x})$ and (ii) $\mathbf{x}$ is (IC) given $(\mathbf{p}, \mathbf{q})$. For every market outcome $m = (\mathbf{p}, \mathbf{x}, \mathbf{q})$, define *the induced allocation* for every buyer type:

$$Q_m(\theta) \triangleq \mathbf{q}(\mathbf{x}(\theta)).$$

Similarly, let U_m be the *induced consumer payoff* in m:

$$U_m(\theta) \triangleq V^B(m|\theta).$$

The total surplus in the market outcome m is given by the average consumer value of all purchased goods:

$$TS(m) \triangleq \int_{\theta:\mathbf{x}(\theta)\in(v^l,\theta)} Q_m(\theta)\theta f(\theta)d\theta + \int_{\theta:\mathbf{x}(\theta)\leq v^l} \left(Q_m(\theta)\theta + (1 - Q_m(\theta))v^l\right) f(\theta)d\theta.$$

Proof of Proposition 4. Neither of the results in Section Appendix B rely on $D_m(1) = 1$. In particular, from Lemma 7, in any market outcome with positive sales, there is a positive mass of consumers visiting outlet locations. From Proposition 6, a sorting equilibrium m admits an outlet threshold $\hat{x}$. By Lemma 8, the inventory quality at the outlet threshold $\mathbf{q}(\hat{x}) > 0$, given total sales are positive $S_m(0) > 0$. And by Lemma 6, locations $[\hat{x}, 1]$ hold the same inventory quality of $\mathbf{q}(\hat{x})$.

Lemma 17. *In every sorting equilibrium $m = (\mathbf{p}, \mathbf{x}, \mathbf{q})$ with positive sales:*

(i) Q_m is increasing

(ii) and every $\theta > v^l$:

$$U_m(\theta) = U_m(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} Q_m(s)ds,$$

where

$$\hat{\theta} = \sup\{\theta : \mathbf{p}(\mathbf{x}(\theta)) \leq v^l\}.$$

Proof. As $\mathbf{q}(\hat{x}) > 0$, all consumer types in $(v^l, v^h]$ receive a strictly positive payoff (the proof is analogous to one of Proposition 6). Then, from (IC), for any two types $\theta, \theta' > v^l$, θ does not have a profitable deviation towards $\mathbf{x}(\theta')$ if and only if:

$$U_m(\theta) \geq U_m(\theta') + Q_m(\theta')(\theta - \theta').$$

As the share of consumers searching at outlet locations is strictly positive, $\hat{\theta} > v^l$. Then, using the standard argument, we obtain that Q_m agrees with IC only if Q_m is increasing on $[\hat{\theta}, v^h]$ and consumer's equilibrium payoff satisfies the envelope formula:

$$U_m(\theta) = U_m(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} Q_m(s)ds.$$

□

Lemma 18. *In every sorting equilibrium with positive sales:*

(i) if $\theta < \hat{\theta}$, then $\mathbf{x}(\theta) \geq \hat{x}$

(ii) $\mathbf{x}$ is decreasing on $(\hat{\theta}, v^h]$

Proof. **Part** (i). Suppose not, and there exists $\tilde{\theta} < \hat{\theta}$, such that $\mathbf{x}(\theta) < \hat{x}$. By definition of $\hat{x}$, $\tilde{\theta}$ visits a non-outlet location $\mathbf{p}(\mathbf{x}(\tilde{\theta})) > v^l$.

By definition of $\hat{\theta}$, either $\hat{\theta}$ visits an outlet location with inventory quality $\mathbf{q}(\hat{x})$, or there exists θ' arbitrarily close to $\hat{\theta}$ searching at such location. As $\mathbf{p}(\mathbf{x}(\tilde{\theta})) > v^l$, due to IC, $Q_m(\tilde{\theta}) > \mathbf{q}(\hat{x})$. But then we obtain a contradiction with monotonicity of Q_m from Lemma 17.

Part (*ii*). By definition of $\hat{\theta}$, all consumer types above $\hat{\theta}$ shop at non-outlet locations contained in $(0, \hat{x})$. Suppose by a way of contradiction that there exist $\theta_1 > \theta_2 > \hat{\theta}$, such that $\hat{x} > \mathbf{x}(\theta_1) > \mathbf{x}(\theta_2)$. By Lemma 6 (N-i), $(1 - \mathbf{q}(\mathbf{x}(\theta_1)))S_m(\mathbf{x}(\theta_1)) = (1 - \mathbf{q}(\mathbf{x}(\theta_2)))S_m(\mathbf{x}(\theta_2))$. Hence, to satisfy the monotonicity condition for Q_m, it must be that a zero mass of consumers shop at locations in $[\mathbf{x}(\theta_2), \mathbf{x}(\theta_2)]$. This is only possible if a non-empty subset of consumer types in (θ_2, θ_1) visits either locations in $(0, \mathbf{x}(\theta_1))$, or locations in $(\mathbf{x}(\theta_2), 1)$.

Either way, we may find some consumer types types $\theta_1' > \theta_2'$, for whom $\mathbf{x}(\theta_1') > \mathbf{x}(\theta_2')$ and there is a non-trivial mass of consumers shopping between their visited locations $(\mathbf{x}(\theta_2'), \mathbf{x}(\theta_1'))$. Bu then $Q_m(\theta_1') < Q_m(\theta_2')$, violating monotonicity of Q_m.

□

Lemma 18 implies the main part of Proposition 4. To show the additional part, I establish that a sorting equilibrium effectively pins down the induced allocation through the outlet shopper $\hat{\theta}$. The construction of the induced allocation is analogous to the construction of $\tilde{\mathbf{q}}_{(D,\gamma,\hat{x})}$ in Lemma 10. Formally, define $Q^{\hat{\theta}} : \Theta \to [0, 1]$ as follows:

$$Q^{\hat{\theta}}(\theta) = \begin{cases} \dfrac{W\left(\frac{\pi}{1-\pi}\exp\left[\frac{\pi}{1-\pi} - \frac{1-F(\theta)}{F(\hat{\theta})(1-\lambda(F(\hat{\theta}),0))}\right]\right)}{1 + W\left(\frac{\pi}{1-\pi}\exp\left[\frac{\pi}{1-\pi} - \frac{1-F(\theta)}{F(\hat{\theta})\left(1-\lambda(F(\hat{\theta}),0)\right)}\right]\right)}, \text{ if } \theta < \hat{\theta} \\ \lambda(F(\hat{\theta}, 0), \text{ if } \theta \geq \hat{\theta}. \end{cases}$$

Lemma 19. *Suppose $m = (\mathbf{p}, \mathbf{x}, \mathbf{q})$ is a sorting equilibrium with positive sales admitting a threshold outlet shopper $\hat{\theta}$. Then $Q_m(\theta) = Q^{\hat{\theta}}(\theta)$, for all $\theta > v^l$.*

Proof. The statement of the lemma follows from Lemma 10, Lemma 11 and Lemma 12. □

In any sorting equilibrium with positive sales, $\hat{\theta} > v^l$. By Lemma 19, the induced quality allocation for any type $\theta > v^l$, is given by $Q^{\hat{\theta}}(\theta)$.

Suppose $\hat{\theta} < v^h$, and define

$$\nu^B(\theta|\hat{\theta}) \triangleq \begin{cases} \lambda\left(F(\hat{\theta}), 0\right)(\theta - v^l), \text{ if } \theta \in [\hat{\theta}, v^l] \\ \lambda\left(F(\hat{\theta}), 0\right)(\hat{\theta} - v^l) + \int_{\hat{\theta}}^{\theta} Q^{\hat{\theta}}(s)ds, \text{ if } \theta \in (\hat{\theta}, v^h]. \end{cases}$$

For types $\theta \leq \hat{\theta}$, $V^B(m|\theta) = \nu^B(\theta|\hat{\theta})$, since all these type search at outlets, where the price is exactly v^l (by Proposition 6). For types $\theta > \hat{\theta}$, $V^B(m|\theta) = \nu^B(\theta|\hat{\theta})$ by Lemma 17.

Similarly, define

$$\nu^{TS}(\hat{\theta}) \triangleq \int_{\hat{\theta}}^{v^h} \theta Q^{\hat{\theta}}(\theta) f(\theta) d\theta + \int_{v^l}^{\hat{\theta}} \left(\theta\lambda\left(F(\hat{\theta}),0\right) + v^l\left(1-\lambda\left(F(\hat{\theta}),0\right)\right)\right) f(\theta) d\theta,$$

then $TS(m) = \nu^{TS}(\hat{\theta})$ from the definition of $\hat{\theta}$. This completes the proof. □

Appendix I Vintage-Based Pricing

I now provide details for the vintage-based pricing model.

The vintage-based sorting equilibrium requires both vintage distribution and inventory quality to be in a steady state.

The total outflow of products with vintages in $(x_1, x_2]$ is: the purchases of these vintages, and the mass of vintage x_2. The total inflow of goods equals the mass of products of vintage x_1. Then, the total stock at vintages in $(x_1, x_2]$ stays the same when:

$$\int_{y\in[x_1,x_2]} \sigma(y)\left[\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \le v^h\} + (1-\mathbf{q}(y))\mathbb{1}\{\mathbf{p}(y) \le v^l\}\right] dy + \mu(x_2) = \mu(x_1) \tag{27}$$

Similarly, the mass of high-quality goods of vintages $(x_1, x_2]$ is preserved when:

$$\int_{y\in[x_1,x_2]} \sigma(y)\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \le v^h\} dy + \mathbf{q}(x_2)\mu(x_2) = \mathbf{q}(x_1)\mu(x_1) \tag{28}$$

We now say that $(\mu, \mathbf{q})$ is sustained by $(\mathbf{p}, \sigma)$ on $Y \subseteq X$ if both Equation (27) and Equation (28) hold for each $(x_1, x_2] \subseteq Y$. Say that $(\mu, \mathbf{q})$ is sustained by $(\mathbf{p}, \sigma)$ if it is sustained on $[0, 1]$.

Payoffs. The payoffs in this alternative formulation of the model remain the same. They are described by the same functions as in the benchmark model. Consumer payoff in a vintage-based market outcome $(\mathbf{p}, \sigma, \mu, \mathbf{q})$ only depends on consumer strategy, prices, and inventory quality and is given by $V^B(\mathbf{p}, \sigma, \mathbf{q})$. The seller's payoff only depends on the distribution of vintages through the rate of disposal. It is given by $V^S(\mathbf{p}, \sigma, \mathbf{q}, \mu(1))$.

Vintage-Based Sorting Equilibrium. We adjust the sorting equilibrium definition to account for the endogeneity of stock distribution across vintages. Say that a vintage-based market outcome $w = (\mathbf{p}, \sigma, \mu, \mathbf{q})$ is a *a vintage-based sorting equilibrium* if both stock distribution and inventory quality $(\mu, \mathbf{q})$ are sustained by prices and consumer strategy $(\mathbf{p}, \sigma)$, and σ maximizes consumer payoff $V^B(\mathbf{p}, \sigma, \mathbf{q})$ for the given $(\mathbf{p}, \mathbf{q})$.

Vintage-Based Sorting Equilibrium. We adjust the sorting equilibrium definition to account for the endogeneity of stock distribution across vintages. Say that a market outcome

$m = (\mathbf{p}, \sigma, \mu, \mathbf{q})$ is a *a vintage-based sorting equilibrium* if (i) both stock distribution and inventory quality $(\mu, \mathbf{q})$ are sustained by prices and consumer strategy $(\mathbf{p}, \sigma)$, and (ii) σ maximizes consumer payoff $V^B(\mathbf{p}, \sigma, \mathbf{q})$ for the given $(\mathbf{p}, \mathbf{q})$.

Proof of Theorem 2. Consider a vintage-based market-outcome $w = (\mathbf{p}, \sigma, \mu, \mathbf{q})$, and a market outcome $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ with disposal rate $\gamma = \mu(1)$. We verify that w is a vintage-based sorting equilibrium if and only if m is a sorting equilibrium.

Part (ii), consumer optimality, is the same across the two notions of equilibrium by definition. We need only show that $(\mu, \mathbf{q})$ is sustained by $(\mathbf{p}, \sigma)$ in w if and only if $\mathbf{q}$ is sustained by $(\mathbf{p}, \sigma, \gamma)$ in m.

By definition, $(\mu, \mathbf{q})$ is sustained by $(\mathbf{p}, \sigma)$ in w, whenever for every $x \in [0, 1]$:

$$\mu(x) = \mu(1) + \int_{y \in [x,1)} \sigma(y) \left[\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\} + (1 - \mathbf{q}(y))\mathbb{1}\{\mathbf{p}(y) \leq v^h\}\right] dy$$

and

$$\mathbf{q}(x)\mu(x) = \mu(1)\mathbf{q}(1) + \int_{y \in [x,1)} \sigma(y)\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\}dy$$

.

And $\mathbf{q}$ is sustained in $(\mathbf{p}, \sigma, \gamma)$ in m whenever for every $x \in [0, 1)$:

$$\mathbf{q}(x)S_m(x) = \gamma\mathbf{q}(1-) + \int_{y \in [x,1)} \sigma(y)\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\}dy,$$

where

$$S_m(x) = \gamma + \int_{y \in (x,1)} \sigma(y) \left[\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \leq v^h\} + (1 - \mathbf{q}(y))\mathbb{1}\{\mathbf{p}(y) \leq v^h\}\right] dy, \forall x \in (0, 1)$$

Note that we can replace $\gamma\mathbf{q}(1-)$ with $\gamma\mathbf{q}(1)$ in the above: if $\gamma > 0$, then $\mathbf{q}(\cdot)$ can only be sustained if it is continuous at 1 by Lemma 5.

If we take $\gamma = \mu(1)$, the two systems are equivalent. For every $x \in (0, 1)$, the downstream sales at location x in m coincide with the mass of goods having vintage x in w.

□

Appendix J Omitted Proofs for Section 6.2

Proof of Proposition 5. Fix some sorting equilibrium $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$. By Proposition 6, it is a $\hat{x}$-threshold market outcome for some outlet threshold $\hat{x}$.

The seller payoff in a market outcome m satisfies:

$$V^S(m) = TS(m) - V^B(m).$$

From Lemma 9, the total surplus in a sorting equilibrium m with positive total sales equals:

$$\begin{aligned} TS(m) &= S_m(0)(\pi v^h + (1-\pi)v^l) - \gamma \mathbf{q}(\hat{x})(v^h - v^l) - \gamma(\kappa + v^l) \\ &= \frac{\int_{y\in[\hat{x},1)} dD(y)(1-\mathbf{q}(\hat{x}))}{1-\pi}(\pi v^h + (1-\pi)v^l) - \gamma \mathbf{q}(\hat{x})(v^h - v^l) - \gamma(\kappa + v^l) \end{aligned}$$

where the second equality is due to Lemma 6 (N-i). By Lemma 12, if D admits finitely many discontinuities at non-outlet locations, then $\mathbf{q}(\hat{x}) > \lambda\left(\int_{y\in[\hat{x},1)} dD(y), \gamma\right)$, so that the total surplus at m satisfies:

$$\begin{aligned} TS(m) < & \frac{\int_{y\in[\hat{x},1)} dD(y)\left(1 - \lambda(\int_{y\in[\hat{x},1)} dD(y), \gamma)\right)}{1-\pi}(\pi v^h + (1-\pi)v^l) \\ & - \gamma\lambda\left(\int_{y\in[\hat{x},1)} dD(y), \gamma\right)(v^h - v^l) - \gamma(\kappa + v^l). \end{aligned}$$

For consumer payoff, we may consider two cases, depending on whether outlet locations are visited. **Case 1: Outlets are visited**. If outlets are visited, then the consumer payoff is at least

$$V^B(m) \geq \mathbf{q}(\hat{x})(v^h - v^l).$$

And the seller's payoff is bounded from above by:

$$\begin{aligned} V^S(m) < & \frac{\int_{y\in[\hat{x},1)} dD(y)\left(1 - \lambda\left(\int_{y\in[\hat{x},1)} dD(y), \gamma\right)\right)}{1-\pi}(\pi v^h + (1-\pi)v^l) \\ & - (\gamma+1)\lambda\left(\int_{y\in[\hat{x},1)} dD(y), \gamma\right)(v^h - v^l) - \gamma(\kappa + v^l). \end{aligned}$$

The seller could achieve the payoff on the right-hand-side in a sorting equilibrium that admits no atoms and has share $\int_{y\in[\hat{x},1)} dD(y)$ visiting outlet locations.

For instance, we may construct a sorting equilibrium $m_1 = (\mathbf{p}_1, D_1, \mathbf{q}_1,)$ as follows. Let consumer strategy to be uniform $D_1(x) = x, \forall x$. Then, we can find the outlet threshold threshold as $\hat{x}_1 = 1 - \int_{y\in[\hat{x},1)} dD(y)$. From Lemma 10, we can find a sustained inventory quality: $\mathbf{q}_1(x) = \tilde{\mathbf{q}}_{(D_1,\gamma,\hat{x}_1)}$. And then, can define prices $\mathbf{p}_1$ to ensure consumer indifference between all locations. At m_1, the inventory quality at $\hat{x}_1$ is exactly $\lambda\left(\int_{y\in[\hat{x},1)} dD(y), \gamma\right)$ (by

Lemma 12).

Case 2: Outlets are not visited. In this case, the seller's payoff is at most:

$$\begin{aligned} V^S(\mathbf{p}, D, \mathbf{q}, \gamma) \leq\ & TS(\mathbf{p}, D, \mathbf{q}, \gamma) \\ <\ & \frac{\int_{y\in[\hat{x},1)} dD(y) \left(1 - \lambda(\int_{y\in[\hat{x},1)} dD(y), \gamma)\right)}{1-\pi}(\pi v^h + (1-\pi)v^l) \\ & - \gamma\lambda\left(\int_{y\in[\hat{x},1)} dD(y), \gamma\right)(v^h - v^l) - \gamma(\kappa + v^l). \end{aligned}$$

Similarly, the seller could achieve the payoff on the right-hand-side of the above with some atomless consumer strategy. □

A Online Appendix

This section includes additional results omitted in the main Appendix of the paper. Section OA1 formally describes the two-store model, which was an illustration in the Introduction of the paper.

Section OA2 constructs the sorting equilibrium for the model with multiple qualities.

OA1 Two-Store Model

This section formalizes a two-store model, which was used as an illustration in the Introduction of the paper. It derives the key comparative statics results summarized in **??** in the Introduction.

There are two stores: a flagship and an outlet. Each store holds a continuum of products of mass 1. The goods have binary value as in Section 2. The share of high-value goods at store $i \in \{f, o\}$ is denoted as q^i. Time is discrete and runs over an infinite horizon, $t \in \{1, 2, \ldots\}$. Each period, a mass $\lambda \in (0, 1)$ of short-lived consumers arrives at the market. Flagship charges a full price $p^f > v^l$, whereas the outlet sells goods at a markdown $p^o = v^l$.
Consumer Behavior. Upon visiting the store, each consumer is matched to a single product at random. The probability of getting matched to a high-value good at location i is q^i. Each product is matched to at most one consumer. Consumers choose between the two stores given prices and shares of high-value goods. Let σ denote the share of consumers who choose the flagship store. Consumers select their shopping strategy σ to maximize the expected payoff given the prices and the inventory quality:

$$V^B(p^f, \sigma, q^f, q^o) = \sigma q^f (v^h - p^f) + (1 - \sigma) q^o (v^h - v^l).$$

Quality Composition Evolution. The stock reallocation is a discrete analog of the continuous model. Inventory is reallocated downstream from production to the flagship to the outlet, to maintain both stores at their full capacity.

Suppose at the beginning of period t, the proportion of high-value goods at each store $i \in \{f, o\}$ is given by q_t^i. Consider the outlet first. At the outlet, consumers purchase any product type they find. Therefore, total sales at the outlet in any given period equal its consumer flow $(1 - \sigma)\lambda$, with a share q_t^o of these sales being of high value. To replenish the outlet, the seller ships inventory from the flagship equal to the total outlet sales, $\lambda(1 - \sigma)$. The share of high-value goods in the shipments is the proportion of high-value goods in the flagship's after-sales remaining inventory, denoted $q_{t,a}^f$. Thus, the total change in the mass of high-value items[25] at the outlet is

$$\Delta q_t^o = q_{t,a}^f \lambda(1-\sigma) - q_t^o \lambda(1-\sigma).$$

Next, consider the evolution of the inventory quality at the flagship store. Consumers only purchase high-value goods there. The total mass of purchases at the flagship equals $\lambda\sigma q_t^f$, the mass of consumers who find a high-value product. The total remaining stock after purchases is then $1-\lambda\sigma q_t^f$, while the remaining mass of high-value goods is $q_t^f(1-\lambda\sigma)$. The resulting after-sales proportion of high-value, $q_{t,a}^f$ is equal to

$$q_{t,a}^f = q_t^f \frac{(1-\lambda\sigma)}{1-\lambda\sigma q_t^f}.$$

The flagship gets restocked to full capacity once consumer purchases and shipments to the outlet are complete. The total mass of new inventory ordered from the production plant to the flagship equals the mass of total sales at both stores in period t, which is $q_t^f\lambda\sigma + \lambda(1-\sigma)$. A fraction π of these new items is of high value. Hence, the change in the flagship's share of high-value items is given by

$$\Delta q_t^f = \pi(q_t^f \lambda\sigma + \lambda(1-\sigma)) - \lambda\sigma q_t^f - q_{t,a}^f \lambda(1-\sigma).$$

Sorting Equilibrium. The inventory quality (q^f, q^o) is *sustained by consumer strategy* σ[26] when the proportion of high-value goods at both stores remains constant over time: $\Delta q_t^i = 0$ for $q_t^i = q^i$.

Flagship price p^f, consumer strategy σ, and the inventory quality (q^f, q^o) form *a sorting* if (i) (q^f, q^o) is sustained by consumer strategy σ and (ii) σ is consumer-optimal given (p^f, q^f, q^o).

OA1.1 Sorting Equilibria in a Two-Store Model

First, I fix the consumer strategy σ and analyze the inventory quality it sustains. Lemma 20 shows that each consumer st

Lemma 20. *If $\sigma = 1$, then any $(q^f, q^o) \in [0,1]^2$, such that $q^f = 0$ is sustained by σ. For every $\sigma < 1$, there exists a unique inventory quality, $(\mathbf{q}^f(\sigma), \mathbf{q}^o(\sigma))$ it sustains. Moreover,*

[25] Given the stock of either store is normalized to one, the change in the proportion of high-value goods at any store coincides with the change of their mass.

[26] We can skip the prices from the definition, as we already fixed them above v^l for the flagship and at v^l for the outlet.

(i) $\mathbf{q}^f(\sigma) > \mathbf{q}^o(\sigma) > 0$

(ii) both $\mathbf{q}^f(\cdot)$ *and* $\mathbf{q}^o(\cdot)$ *are decreasing, with* $\mathbf{q}^f(0) = \mathbf{q}^o(0) = \pi$*,*

(iii) and $\mathbf{q}^f(\cdot)/\mathbf{q}^o(\cdot)$ *is strictly increasing.*

Proof. By definition, (q^f, q^o) is sustained by σ whenever:

$$\lambda(1-\sigma)\frac{(1-\lambda\sigma)q^f}{1-\lambda\sigma q^f} - \lambda(1-\sigma)q^o = 0, \tag{29}$$

$$\pi(q^f\lambda\sigma + \lambda(1-\sigma)) - \lambda\sigma q^f - \frac{(1-\lambda\sigma)q^f}{1-\lambda\sigma q^f}\lambda(1-\sigma) = 0. \tag{30}$$

Step 1: $\sigma = 1$. In this case, Equation (29) is satisfied for any q^o, q^f. And Equation (29) at $\sigma = 1$ becomes:

$$\pi q^f \lambda - \lambda q^f = 0.$$

As $\pi < 1$, the above is only satisfied when $q^f = 0$.
Step 2: $\sigma < 1$. In this case, Equation (29) reduces to:

$$q^o = \frac{(1-\lambda\sigma)q^f}{1-\lambda\sigma q^f}$$

That is, the outlet's inventory quality coincides with the flagship's after-sales average value of goods. In addition, if (q^f, q^o) is sustained by σ, then $q^o < q^f$.

To solve for q^f which can be sustained by σ, define $\Psi(q^f, \sigma)$:

$$\Psi(\sigma, q^f) = \pi(\sigma q^f + (1-\sigma)) - q^f\sigma - \delta q^f(1/\lambda - \sigma) - \frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\lambda\sigma}(1-\sigma)$$

Then, q^f is sustained by σ whenever $\Psi(q^f, \sigma) = 0$. I now show that there exists a unique such q^f for every σ. To that end, it is sufficient to show that $\Psi(\cdot, \sigma)$ is decreasing in q^f for every σ and $\Psi(\cdot, \sigma)$ changes its sign at some interior q^f.

Existence. $\Psi(0, \sigma) = \pi(1-\sigma) \geq 0$, where the inequality is strict if and only if $\sigma < 1$. On the other hand, $\Psi(\pi, \sigma)$ is:

$$\begin{aligned}
\Psi(\pi, \sigma) &= \pi(1-\sigma) - \sigma(1-\pi)\pi - (1-\sigma)\frac{\pi(1-\lambda\sigma)}{1-\pi\sigma\lambda} \\
&= \frac{\pi(1-\pi)\lambda\sigma(1-\sigma)}{1-\pi\sigma\lambda} - \sigma(1-\pi)\pi \\
&= -\pi(1-\pi)\sigma\frac{\lambda\sigma(1-\pi) + (1-\lambda)}{1-\pi\sigma\lambda} \leq 0
\end{aligned}$$

Hence, for every $\sigma < 1$ there exist $q^f \in [0, 1]$, such that $\Psi(q^f, \sigma) = 0$ (by the Intermediate Value Theorem due to continuity of Ψ in q^f).

Uniqueness. Now, let me verify that $\Psi(q^f, \sigma)$ is decreasing in q^f:

$$\partial_{q^f}\Psi(q^f, \sigma) = -\sigma(1-\pi) - \frac{(1-\sigma)(1-\lambda\sigma)}{(1-q^f\sigma\lambda)^2} < 0$$

Hence, an intersection with 0 is unique for every σ. I can denote such an intersection as $\mathbf{q}^f(\sigma)$. Given the uniqueness of the flagship inventory quality that can be sustained by σ, a sustained q^o is also unique. To summarize, for every $\sigma < 1$, there is a unique inventory quality $(\mathbf{q}^f(\sigma), \mathbf{q}^o(\sigma))$ it sustains. In addition, $\mathbf{q}^f(\sigma) > \mathbf{q}^o(\sigma) > 0$, for ever $\sigma < 1$, and And $\mathbf{q}^f(0) = \mathbf{q}^o(0) = \pi$.

Step 3: *comparative statics of* $\mathbf{q}^f(\sigma)$. By an Implicit Function Theorem, we have:

$$\partial_\sigma \mathbf{q}^f(\sigma) = -\partial_\sigma\Psi(\mathbf{q}^f(\sigma), \sigma)/\partial_{q^f}\Psi(\mathbf{q}^f(\sigma), \sigma)$$

In Step 2, we showed $\partial_{q^f}\Psi(\mathbf{q}^f(\sigma), \sigma) < 0$. Then, the sign of $\partial_\sigma \mathbf{q}^f(\sigma)$ is determined by the sign of $\partial_\sigma\Psi(\mathbf{q}^f(\sigma), \sigma)$. I now show that $\partial_\sigma\Psi(q^f, \sigma) < 0$ for every $q^f \leq \pi$.

$$\begin{aligned}
\partial_\sigma\Psi(q^f, \sigma) &= -\pi - q^f(1-\pi) + (1+\lambda-2\lambda\sigma)\frac{q^f}{1-\sigma q^f\lambda} \\
&\quad - \lambda(1-\sigma)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f\lambda}\right)^2 \\
&= -\pi - q^f(1-\pi) + 2\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f\lambda} - \left(\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f\lambda}\right)^2 \\
&\quad - (1-\lambda)\frac{q^f}{1-\sigma q^f\lambda} + (1-\lambda)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f\lambda}\right)^2
\end{aligned}$$

Using

$$\begin{aligned}
&- (1-\lambda)\frac{q^f}{1-\sigma q^f\lambda} + (1-\delta)(1-\lambda)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f\lambda}\right)^2 \\
&= (1-\lambda)q^f\left(\frac{1}{1-\sigma q^f\lambda}\right)^2\left[-(1-q^f\sigma\lambda) + q^f(1-\lambda\sigma)\right] \leq 0
\end{aligned}$$

we can bound $\partial_\sigma\Psi(q^f, \sigma)$ by:

$$\partial_\sigma\Psi(q^f, \sigma) \leq -\pi - q^f(1-\pi) + 2\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f\lambda} - \left(\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f\lambda}\right)^2.$$

Additionally, as $\frac{(1-\lambda\sigma)q^f}{1-\sigma\lambda q^f} \leq q^f$ and $2x - x^2$ increasing in x for $x \leq 1$, we can further bound the above:

$$\partial_\sigma \Psi(q^f, \sigma) \leq -\pi - q^f(1-\pi-\delta) + 2q^f - \left(q^f\right)^2 \tag{31}$$

Now, I show that the Expression (31) that bounds $\partial_\sigma \Psi(q^f, \sigma)$ is increasing in q^f. Differentiating it with respect to q^f, we get:

$$\begin{aligned}\partial_{q^f}\left(-\pi - q^f(1-\pi) + 2q^f - \left(q^f\right)^2\right) &= 1 + \pi - 2(1-\delta)q^f \\ &\geq 1 + \pi - 2\pi \\ &= (1-\pi) > 0\end{aligned}$$

Hence, we can bound $\partial_\sigma \Psi(q^f, \sigma)$ by plugging $q^f = \pi$ in the Expression (31):

$$\begin{aligned}\partial_\sigma \Psi(q^f, \sigma) &\leq -\pi - \pi(1-\pi) + 2\pi - \pi^2 \\ &\leq -\pi - \pi(1-\pi) + 2\pi - \pi^2 = 0.\end{aligned}$$

Then, $\partial_\sigma \Psi(q^f, \sigma) \leq 0$ and $\partial_\sigma \Psi(q^f, \sigma) < 0$ for $q^f < \pi$.

Step 4: *comparative statics for the sorting precision* $\mathbf{q}^f(\sigma)/\mathbf{q}^o(\sigma)$.

Differentiating the sorting precision with respect to σ, we get:

$$\begin{aligned}\partial_\sigma \mathbf{q}^f(\sigma)/\mathbf{q}^o(\sigma) = \partial_\sigma \frac{1-\lambda\sigma\mathbf{q}^f(\sigma)}{1-\lambda\sigma} &= \frac{-\lambda\mathbf{q}^f(\sigma)(1-\lambda\sigma) + \lambda(1-\lambda\sigma\mathbf{q}^f(\sigma))}{(1-\lambda\sigma)^2} \\ &= \frac{\lambda(1-\mathbf{q}^s(\sigma)) - \lambda\sigma\partial_\sigma\mathbf{q}^f(\sigma)}{(1-\lambda\sigma)^2} > 0\end{aligned}$$

where the we use $\partial_\sigma \mathbf{q}^f(\sigma) < 0$ by Step 3. In addition, as $\mathbf{q}^f(\cdot)$ is decreasing but $\mathbf{q}^f(\cdot)/\mathbf{q}^o(\cdot)$ is increasing, then $\mathbf{q}^o(\cdot)$ is decreasing. This completes the proof of the lemma. □

Convergence to the Steady State. I now discuss the convergence of the inventory quality to the sustained steady state to interpret the sorting equilibrium.

The inventory quality evolution described above assumes a constant consumer strategy. To interpret, suppose consumers do not see the time of their arrival or the current inventory quality. In equilibrium, they correctly anticipate the long-run inventory quality at the two stores, which dominates their beliefs.

To make sure this interpretation is valid, we must verify that for a given σ, the inventory quality converges to a unique steady state. I illustrate convergence to the steady state in Figure 8. To establish convergence, note that Δq_t^f only depends on q_t^f and is decreasing in it. Hence, whenever $q_t^f > (<)\mathbf{q}^f(\sigma)$, $\Delta q_t^f < (>)0$. The inventory quality at the flagship is pushed towards the steady state. In turn, for a given q_t^f, Δq_t^o is decreasing in q_t^o. Hence, the steady state is stable.

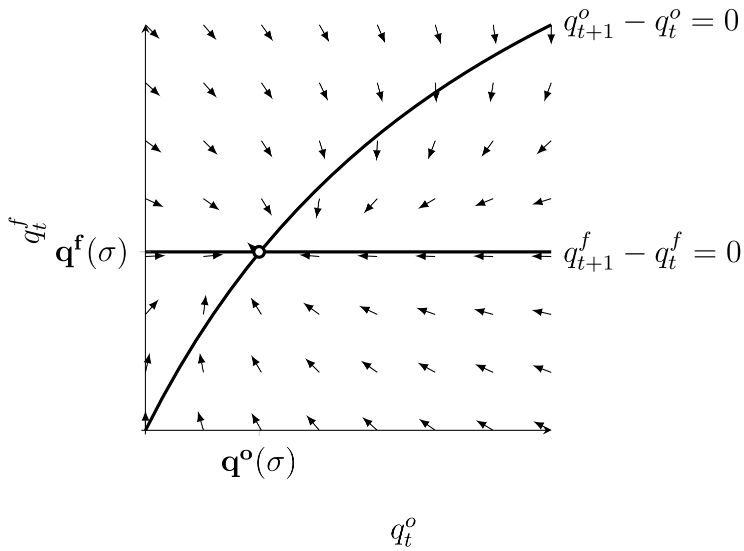

Figure 8: Phase Diagram

Interior Equilibrium. We can now formulate the main result for the two-store model: comparative statics of the interior sorting equilibrium with respect to the flagship price p^f.

Proposition 7. *For every flagship price $p^f \in (v^l, v^h)$, there exists at most one interior sorting equilibrium (p^f, σ, q^f, q^o). Moreover, in this equilibrium, if p^f increases*

(i) the customer share of the flagship store σ rises,

(ii) the inventory quality at both stores (q^f, q^o) gets worse,

(iii) the sorting precision q^f/q^o rises,

(iv) and the total steady-state per-period sales $\lambda[\sigma q^f + (1-\sigma)]$ decrease.

Proof. By Lemma 20, for every interior σ, there exists a unique inventory quality that it sustains. In addition, at any such equilibrium, consumers must be indifferent between the two stores, which implies:

$$\frac{\mathbf{q}^f(\sigma)}{\mathbf{q}^o(\sigma)} = \frac{v^h - v^l}{v^h - p^f} \tag{IND}$$

By Lemma 20, the sorting precision $\frac{\mathbf{q}^f(\sigma)}{\mathbf{q}^o(\sigma)}$ is strictly increasing in σ. Then, for every price p^f, there exists at most one σ, where Equation (IND) is true. Consequently, by Lemma 20, for every p^f there exists at most one interior sorting equilibrium. Moreover, as p^f becomes larger, the sorting precision must rise, which requires σ to increase in the interior equilibrium. Part (i) follows. Parts $(ii), (iii)$ then follow from Lemma 20. Finally, the steady-state sales at this equilibrium are given by

$$\lambda[\sigma \mathbf{q}^f(\sigma) + (1-\sigma)],$$

and are decreasing in σ.

□

OA2 Sorting Equilibrium Construction for Multiple Value Types

For any sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$, and define a_m be the lowest-always-purchased value:

$$a_m \triangleq \min_{n \geq i \geq 1} \left\{ i : \int_{y:\mathbf{p}(y) > v^i} \sigma(y) dy = 0 \right\}$$

and define e_m be the lowest-ever-purchased value:

$$e_m \triangleq \min_{n \geq i \geq 1} \left\{ i : \int_{y:\mathbf{p}(y) \leq v^i} \sigma(y) dy > 0 \right\}$$

Proposition 8. *Consider two sorting equilibria $m_1, m_2 \in \mathcal{E}$ with positive sales and the same disposal rate $\gamma \geq 0$. If they induce the same lowest-ever-purchased value, i.e. $e_{m_1} = e_{m_2}$, and the same consumer surplus, i.e. $V^B(m_1) = V^B(m_2)$, then they also induce the sale seller profit, i.e. $V^S(m_1) = V^S(m_2)$.*

Proof. To prove Proposition 8, I characterize the construction of the sorting equilibrium with multiple qualities. First, I show that the choice of the induced consumer payoff pins down the lowest-always-purchased value. Effectively, the seller picks the highest price through her choice of the consumer surplus.

Lemma 21. *If a sorting equilibrium m induces consumer payoff $CS \geq 0$ and has the lowest-purchased-value a_m, then:*

$$\sum_{k \geq a^m} \pi(k)(v^k - v^{a_m - 1}) \geq CS \geq \sum_{k \geq a^m} \pi(k)(v^k - v^{a_m}),$$

Proof. Find the earliest visited location

$$\bar{y} = \sup \left\{ y : \int_{x \in (0,y)} \sigma(x) dx = 0 \right\}.$$

Then, it must be that in any right neighborhood of $\bar{y}$, consumers shop with positive probability, and we can find a converging sequence of visited locations $\{z_k\}$ with $z_k \to \bar{y}$, such that the consumer's payoff is given by :

$$CS = \sum_{i=1}^{n} \mathbf{q}(i|z_k)(v^i - \mathbf{p}(z_k))_+$$

a_m is the lowest-always-purchased value, then $\hat{x}_{a_m - 1} > \bar{y}$ and by Proposition 6, $\mathbf{p}(\cdot) \in (v^{j+1}, v^j)$ $[\bar{y}, \hat{x}_{a_m-1}]$, D-a.s.. Then, along the sequence $\{z_k\}$, consumer payoff is bounded by:

$$\sum_{i=1}^{n} \mathbf{q}(i|z_k) \left(v^i - v^{a^m}\right)_+ \leq CS \leq \sum_{i=1}^{n} \mathbf{q}(i|z_k) \left(v^i - v^{a^m - 1}\right)_+$$

By Lemma 5, $\mathbf{q}(\cdot)$ is continuous at $\bar{y}$, then as $z_k \to \bar{y}_j$, $\mathbf{q}(i|z_k) \to \mathbf{q}(i|\bar{y})$. As no locations are visited on $(0, \bar{y})$, they all are outlets D-a.s.. Then, by Lemma 6 (*ii*): $\mathbf{q}(i|\bar{y}) = \mathbf{q}(i|0) = \pi(i)$, and we obtain:

$$\sum_{k \geq a^m} \pi(k) \left(v^k - v^{a^m}\right) \leq CS \leq \sum_{k \geq a^m} \pi(k) \left(v^k - v^{a^m - 1}\right)$$

as required. □

Second, I characterize jumps that the purchase probabilities make when the price crosses the product's potential values.

Lemma 22. *Consider a sorting equilibrium* $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ *with positive total sales and consumer surplus* $CS \geq 0$*. For any* $i \in \{a_m - 1, e_m\}$*, the purchase probability at the threshold location* $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \leq v^i\}$*, jumps by:*

$$\rho_m(\hat{x}_i -) + (1 - \rho_m(\hat{x}^i -)) \frac{\pi(i)}{\sum_{l \leq i} \pi(l)}.$$

Moreover, the expected value of the product of a product conditional on purchase is constant between any two threshold locations $[\hat{x}_i, \hat{x}_{i-1})$ *and must satisfy:*

$$E_m^i = \frac{\rho_m(\hat{x}_i -) E_m^{i+1} + (1 - \rho_m(\hat{x}_i -)) \frac{\pi(i)}{\sum_{l \leq i} \pi(l)} v^i}{\rho_m(\hat{x}_i)} \tag{32}$$

Proof. At any threshold $\hat{x}_i$, the probability of purchase jumps upwards by exactly $\mathbf{q}(i|\hat{x}_i)$, as consumers start purchasing value i. Given Lemma 6 (*ii*), there is no learning about value i relative to any other value $(0, \hat{x}_i)$ which is not purchased on this interval:

$$\frac{\mathbf{q}(i|\hat{x}_i)}{\sum_{l\leq i}\mathbf{q}(l|\hat{x}_i)} = \frac{\mathbf{q}(i|\hat{x}_i)}{1-\rho_m(\hat{x}^i-)} = \frac{\pi(i)}{\sum_{l\leq i}\pi(l)}.$$

Then, we can jump in the purchasing probability at $\hat{x}_i$ must be:

$$\rho_m(\hat{x}_i) = \rho_m(\hat{x}_i-) + \mathbf{q}(i|\hat{x}_i) = \rho_m(\hat{x}_i-) + (1-\rho_m(\hat{x}^i-))\frac{\pi(i)}{\sum_{l\leq i}\pi(l)}.$$

The part about the conditional expected value is similarly implied by Lemma 6 (*ii*).

□

Using the consumer indifference condition between all visited locations, we can derive further restrictions on the equilibrium purchase probabilities at the threshold locations.

Lemma 23. *Consider a sorting equilibrium* $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ *with positive total sales and consumer surplus* $CS \geq 0$*. For any* $i \in \{a_m - 1, e_m\}$*, the purchase probability at the threshold location* $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \leq v^i\}$ *satisfies:*

$$\frac{CS}{\rho_m(\hat{x}_{i-1}-)} = \frac{CS}{\rho_m(\hat{x}_i)} + v^i - v^{i-1},$$
$$\text{where } \rho_m(\hat{x}_i) = \rho_m(\hat{x}_i-) + (1-\rho_m(\hat{x}_i-))\frac{\pi(i)}{\sum_{l\leq i}\pi(l)}$$
$$\text{and } \rho_m(\hat{x}_{a_m-1}-) = \frac{CS}{\frac{\sum_{l\geq a_m}\pi(l)v^l}{\sum_{l\geq a_m}\pi(l)} - v^{a_m-1}}$$

Proof. Lemma 22 implies the following simple identity at any threshold location:

$$\rho(\hat{x}_{i+1})\left(E_m^{i+1} - v^{i+1}\right) = \rho(\hat{x}_{i+1}-)\left(E_m^{i+2} - v^{i+1}\right). \tag{33}$$

Step 1: *indifference condition.* I show that for every $i \leq a_m - 1$ such that $\hat{x}_i < 1$, we must have:

$$CS = \rho_m(\hat{x}_i-)(E_m^{i-1} - v^i),$$
$$\int_{y\in[\hat{x}_i,\hat{x}_{i-1})}\sigma(y)dy > 0.$$

I establish the above by induction.

Initial Iteration: $i = a_m - 1$. If $\hat{x}_{a_m-1} < 1$, then the price is below v^{a_m-1} at some locations in any left neighborhood of $\hat{x}_{a_m-1}$. By definition of the lowest-always-purchased value:

$$\int_{y\in[\hat{x}_{a_m},\hat{x}_{a_m-1})} \sigma(y)dy > 0.$$

The inventory quality is continuous by Lemma 5. Then, the consumer's payoff is at least:

$$\begin{aligned} V^B(\mathbf{p},\sigma,\mathbf{q},\gamma) = CS &\geq \sum_{l\geq a^m} \mathbf{q}(l|\hat{x}_{a_m-1})\left(v^l - v^{a_m-1}\right) \\ &= \rho(\hat{x}_{a_m-1}-)\left(E_m^{a_m} - v^{a_m-1}\right) \end{aligned}$$

Consider the last visited location with the price above v^{a_m-1}:

$$\hat{y}_{a_m-1} = \sup_{x\in[0,\hat{x}_{a_m-1}]} \left\{ \int_{z\in(y,\hat{x}_{a_m-1}]} \sigma(y)dy = 0 \right\}.$$

Then, consumers do not visit locations in $[\hat{y}_{a_m-1}, \hat{x}^{j+1}]$ and $\rho(\hat{y}_{a_m-1}) = \rho(\hat{x}_{a_m-1}-)$. In addition, consumers shop with a positive probability in the right neighborhood of $\hat{y}_{a_m-1}$. If the consumer's shopping strategy is optimal, there is a sequence $\{z_l\}_{l=1}^{\infty}$ converging to $\hat{y}_{a_m-1}$, such that consumers obtain their payoff at each of locations $\{z_l\}_{l=1}^{\infty}$:

$$CS = \rho(z_l)(E_m^{a_m} - \mathbf{p}(z_l)) \leq \rho(z_l)\left(E_m^{a_m} - v^{a_m-1}\right) \underset{z_l\to\hat{y}_{a_m-1}}{\rightarrow} \rho(\hat{x}_{a_m-1}-)\left(E_m^{a_m} - v^{a_m-1}\right)$$

Together, the two bounds imply that:

$$CS = \rho(\hat{x}_{a_m-1}-)\left(E_m^{a_m} - v^{a_m-1}\right)$$

Iteration i. Suppose the statement is true for all $k \geq i+1$, and let us verify that it must then be true for i. If $\hat{x}_i = 1$, then we are done. Otherwise, suppose that $\hat{x}_i < 1$, then the consumer payoff is at least:

$$V^B(\mathbf{p},\sigma,\mathbf{q},\gamma) = CS \geq \rho(\hat{x}_i-)(E_m^{i+1} - v^i)$$

If $\int_{y\in[\hat{x}_{i+1},\hat{x}_i)}^{\hat{x}_i} \sigma(y)dy = 0$, the inventory quality remains the same over an interval $[\hat{x}_{i+1}, \hat{x}_i]$ due to Lemma 6, then:

$$\rho(\hat{x}_i-)\left(E_m^{i+1} - v^{i+1}\right) = \rho(\hat{x}_{i+1})\left(E_m^{i+1} - v^{i+1}\right)$$

Using Equation (33) from Step 1, we have:

$$\rho(\hat{x}_{i+1}-)\left(E_m^{i+2}-v^{i+1}\right)=\rho(\hat{x}_i-)\left(E_m^{i+1}-v^{i+1}\right)<\rho(\hat{x}_i-)\left(E_m^{i+1}-v^{i}\right)\leq CS$$

We obtain a contradiction with the hypothesis of the induction step: $CS=\rho(\hat{x}_{i+1}-)\,(E_m^{i+2}-v^{i+1})$. Hence, some locations at prices between v^i and v^{i+1} are visited: $\int_{y\in[\hat{x}_{i+1},\hat{x}_i)}^{\hat{x}_i}\sigma(y)dy>0$. Similar to the proof in the initial iterative step, consumers visit locations that have a payoff converging to $\rho(\hat{x}_i-)(E_m^{i+1}-v^i)$ from below. This completes the proof by induction.

Step 2. We now combine the two steps. As a_m is the lowest-always-purchased value, then $\mathbf{q}(l|\hat{x}_{a_m})=\mathbf{q}(l|\hat{x}_{a_m}),\forall l$. Conditional on purchase on the interval $(0,\hat{x}_{a_m-1})$, consumer receives payoff:

$$E_m^{a_m}=\frac{\sum_{l\geq a_m}\pi(l)v^l}{\sum_{l\geq a_m}\pi(l)}$$

Using Step 1, the purchase probability at $\hat{x}_{a_m-1}$ satisfies:

$$CS=\rho_m(\hat{x}_{a_m-1}-)\left(E_m^{a_m}-v^{a_m-1}\right)=\rho(\hat{x}_{a_m-1}-)\left(\frac{\sum_{l\geq a_m}\pi(l)v^l}{\sum_{l\geq a_m}\pi(l)}-v^{a_m-1}\right),$$

which implies that $\rho(\hat{x}_{a_m-1}-)$ is as in the statement of the lemma.

For every $i\in\{a_m-1,e_m\}$, the product qualities are purchased with a positive probability, then $\hat{x}_i<1$, and by Step 1:

$$CS=\rho_m(\hat{x}_{i-1}-)\left(E_m^{i}-v^{i-1}\right)=\rho_m(\hat{x}_i-)\left(E_m^{i+1}-v^{i}\right).$$

Using Equation (33), we can replace $\rho_m(\hat{x}_i-)\,(E_m^{i+1}-v^i)$ with $\rho_m(\hat{x}_i)\,(E_m^{i}-v^i)$ in the above to obtain:

$$\frac{CS}{\rho(\hat{x}_i-)}=\frac{CS}{\rho(\hat{x}_i)}+v^i-v^{i-1}.$$

□

Lemma 24. *For any sorting equilibrium $m=(\mathbf{p},\sigma,\mathbf{q},\gamma)\in\mathcal{E}$ with positive total sales and consumer surplus $CS\geq 0$, if consumers purchase all value types with positive probability, i.e. $e_m=1$, then:*

$$\frac{(1-\rho_m(\hat{x}_{e_m}-))}{\sum_{j\leq e_m}\pi(j)}\sum_{i=e^m}^{a_m-1}\left\{\left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}\frac{1-\rho_m(\hat{x}_i-)}{\rho_m(\hat{x}_i-)}\right)+\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}-\frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)}\right]\sum_{l\leq i}\pi(l)\right\}$$

$$= \frac{D(\hat{x}_{e_m})}{1 - D(\hat{x}_{e_m}) + \gamma},$$

and if $e_m > 1$, *then*

$$\frac{(1-\rho_m(\hat{x}_{e_m-1}))}{\sum_{j\leq e_m-1}\pi(j)} \sum_{i=e^m-1}^{a_m-1} \left\{ \left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} \frac{1-\rho_M(\hat{x}_i-)}{\rho_m(\hat{x}_i-)} \right) + \frac{\rho(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} - \frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)} \right] \sum_{l\leq i} \pi(l) \right\}$$
$$= \frac{1}{\gamma}$$

Proof. By Lemma 6, $(1-\rho(x))S_m(x)$ remains constant over any $(\hat{x}_{i+1}, \hat{x}_i)$, in addition since $\partial_x S_m(x) = -\rho_m(x)\sigma(x)$, and we obtain that over $(\hat{x}_{i+1}, \hat{x}_i)$:

$$\partial_x \rho(x) = -\rho_m(x)(1-\rho_m(x))\frac{\sigma(x)}{S_m(x)}. \tag{34}$$

Recall that e_m is the lowest value that is purchased with positive probability, then we have $\sum_{l\leq e_m} \mathbf{q}(l|x)S_m(x)$ remains constant over $[0, \hat{x}_{e_m})$. In addition, Lemma 6 implies there is no relative sorting between any qualities that are not purchased over $(0, \hat{x}_i)$. In particular, the relative inventory quality between qualities below i and qualities below e_m stays constant, and for every $x \in (\hat{x}_{i+1}, \hat{x}_i)$:

$$(1-\rho_m(x)) = \sum_{l\leq e_m} \mathbf{q}(l|x) = \sum_{l\leq e_m} \mathbf{q}(l|x)\frac{\sum_{l\leq i}\pi(l)}{\sum_{j\leq e_m}\pi(j)}.$$

This lets us rewrite Equation (34) as:

$$\partial_x \rho_m(x) = -\rho_m(x)(1-\rho_m(x))^2 \frac{\sigma(x)}{(1-\rho(\hat{x}_{e_m}-))S_m(\hat{x}_{e_m})} \frac{\sum_{j\leq e_m}\pi(j)}{\sum_{l\leq i}\pi(l)}$$
$$-\frac{\sum_{l\leq i}\pi(l)}{\sum_{j\leq e_m}\pi(j)} \frac{\partial_x \rho_m(x)}{\rho_m(x)(1-\rho_m(x))^2} = \frac{\sigma(x)}{(1-\rho_m(\hat{x}_{e_m}-))S_m(\hat{x}_{e_m})}$$

Integrating both sides of the above over $x \in (\hat{x}_{i+1}, \hat{x}_i)$, we get:

$$\frac{\sum_{l\leq i}\pi(l)}{\sum_{j\leq e_m}\pi(j)} \left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho(\hat{x}_{i+1})} \frac{1-\rho_m(\hat{x}_i-)}{\rho(\hat{x}_i-)} \right) + \frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} - \frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)} \right]$$
$$= \frac{D(\hat{x}_i) - D(\hat{x}_{i+1})}{(1-\rho_m(\hat{x}_{e_m}-))S_m(\hat{x}_{e_m})}$$

Summing up the above over all intervals $[\hat{x}_{a_m}, \hat{x}_{a_m} - 1), \ldots, (\hat{x}_{e_m+1}, \hat{x}_{e_m}]$, we get:

$$\frac{(1-\rho_m(\hat{x}_{e_m}-))}{\sum_{j\le e_m}\pi(j)} \sum_{i=e^m}^{a_m-1} \left\{ \left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} \frac{1-\rho_M(\hat{x}_i-)}{\rho_m(\hat{x}_i-)} \right) + \frac{\rho(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} - \frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)} \right] \sum_{l\le i}\pi(l) \right\}$$
$$= \frac{D(\hat{x}_{e_m})}{S_m(\hat{x}_{e_m})} = \frac{D(\hat{x}_{e_m})}{1-D(\hat{x}_{e_m})+\gamma}$$

When $e_m > 1$, then we can similarly sum over $[\hat{x}_{a_m}, \hat{x}_{a_m} - 1), \ldots, (\hat{x}_{e_m}, \hat{x}_{e_m-1}]$ to get:

$$\frac{(1-\rho_m(\hat{x}_{e_m-1}))}{\sum_{j\le e_m-1}\pi(j)} \sum_{i=e^m-1}^{a_m-1} \left\{ \left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} \frac{1-\rho_M(\hat{x}_i-)}{\rho_m(\hat{x}_i-)} \right) + \frac{\rho(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})} - \frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)} \right] \sum_{l\le i}\pi(l) \right\}$$
$$= \frac{D(\hat{x}_{e_m-1})}{S_m(\hat{x}_{e_m-1})} = \frac{1}{\gamma}$$

□

Finally, we derive a general formulation of the irrelevance result for the total surplus.

Lemma 25. *Consider sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ with positive total sales and consumer surplus $CS \ge 0$. If consumers purchase all value types with positive probability, i.e. $e_m = 1$, the total surplus is:*

$$TS(m) = S_m(\hat{x}_{e_m})\,(1-\rho_m(\hat{x}_{e_m}-)) \left[\sum_{i=a_m}^{e_m+1} E_m^i \left(\frac{1}{1-\rho_m(\hat{x}_i)} \frac{\sum_{j\le e_m-1}\pi(j)}{\sum_{j\le i}\pi(i)} - 1 \right) + E_m^{e_m} \right] - E_m^{e_m}\gamma$$

And if some qualities are only cleared through disposal i.e. $e_m > 1$, the total surplus is:

$$TS(m) = \gamma\,(1-\rho_m(\hat{x}_{e_m-1})) \left[\sum_{i=a_m}^{e_m+1} E_m^i \left(\frac{1}{1-\rho_m(\hat{x}_i)} \frac{\sum_{j\le e_m}\pi(j)}{\sum_{j\le i}\pi(i)} - 1 \right) + E_m^{e_m} \right] - E_m^{e_m}\gamma$$

Proof. **Case 1:** $e_m = 1$. The total surplus is:

$$TS(m) = \sum_{i=a_m}^{e_m+1} E_m^i\,(S_m(\hat{x}_i) - S_m(\hat{x}_{i-1})) + E_m^{e_m}(S_m(\hat{x}_{e_m}) - \gamma).$$

By Lemma 6, for every $i \in \{e_m+1, a_m\}$: $(1-\rho(\hat{x}_i))S(\hat{x}_i) = (1-\rho(\hat{x}_{i-1}-))S(\hat{x}_{i-1})$. Then, we can replace $S(\hat{x}_i)$ by:

$$S(\hat{x}_i) = \frac{1-\rho_m(\hat{x}_{i-1}-)}{(1-\rho_m(\hat{x}_i))} S(\hat{x}_{i-1}) = S(\hat{x}_{e_m}) \prod_{j=i}^{e^m-1} \frac{1-\rho_m(\hat{x}_{i+1}-)}{(1-\rho_m(\hat{x}_j))}$$

$$= S(\hat{x}_{e_m}) \frac{1-\rho_m(\hat{x}_{e_m}-)}{1-\rho_m(\hat{x}_i)} \prod_{j=i+1}^{e^m-1} \frac{1-\rho_m(\hat{x}_j-)}{(1-\rho_m(\hat{x}_j))}$$

By Lemma 22, for each i the jump in the purchase probability satisfies:

$$\frac{1-\rho_m(\hat{x}_j-)}{(1-\rho_m(\hat{x}_j))} = \frac{\sum_{l\leq j} \pi(j)}{\sum_{l\leq j} \pi(j)},$$

which implies from the above:

$$S(\hat{x}_i) = S(\hat{x}_{e_m}) \frac{1-\rho_m(\hat{x}_{e_m}-)}{1-\rho_m(\hat{x}_i)} \frac{\sum_{j\leq e_m-1} \pi(j)}{\sum_{j\leq i} \pi(i)}.$$

$$\begin{aligned} TS(m) &= \sum_{i=a_m}^{e_m+1} E_m^i \left(S_m(\hat{x}_i) - S_m(\hat{x}_{i-1})\right) + E_{e_m}(S_m(\hat{x}_{e_m}) - \gamma) \\ &= S_m(\hat{x}_{e_m}) \left(1-\rho_m(\hat{x}_{e_m-1})\right) \left[\sum_{i=a_m}^{e_m+1} E_m^i \left(\frac{1}{1-\rho_m(\hat{x}_i)} \frac{\sum_{j\leq e_m-1} \pi(j)}{\sum_{j\leq i} \pi(i)} - 1 \right) + E_m^{e_m} \right] - E_m^{e_m}\gamma \end{aligned}$$

Case 2: $e_m > 1$ is derived analogously. The only difference is that there is no jump at $\hat{x}_{e_m-1} = 1$. □

Taking stock, Lemma 23 implies that the purchase probabilities at the threshold locations are the same at the two sorting equilibria m_1, m_2 if they have the same consumer payoff CS. By Lemma 22, the conditional expected value of the products is similarly the same. If $e_{m_1} = e_{m_2} = 1$, by Lemma 24, both sorting equilibria have the same downstream sales at $\hat{x}_{e_{m_1}}$ and $\hat{x}_{e_{m_2}}$, respectively. Then, by Lemma 25, the two sorting equilibria have the same total surplus.

If $e_{m_1} = e_{m_2} > 1$, by Lemma 24, then both equilibria have the same downstream sales at $\hat{x}_{e_{m_1}-1} = \hat{x}_{e_{m_2}-1} = 1$. Both equilibria then deliver the same total surplus.

By assumption, they induce the same consumer payoff. As the total surplus is the same across the two sorting equilibria, the seller's payoff must be the same. This concludes the proof.

□