Preliminary and Incomplete

Closing Deals and Tipping Points

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Abstract

I model an uninformed seller selling an object to informed buyers when completing the transaction takes time and effort. The buyer exerts costly effort to speed up the closing stage of the deal. The seller slowly learns about the buyer's enthusiasm for the offered product and reconsiders her pricing strategy as time passes. Knowing this, the buyer can strategically slow down or speed up her work. I show that the dynamics of beliefs, and hence the final prices paid at the closing stage, exhibit tipping points. The seller gradually becomes more pessimistic with time. Occasionally, beliefs jump down abruptly because the less enthusiastic buyer type decides to step up their efforts discontinuously when the seller becomes pessimistic enough. Under some conditions, the market comes to a freeze right before a burst of activity. Actions of lower buyer types are essential to resolving the market freezes and generating tipping points in the closing process.

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1 Introduction

This paper studies selling a good to a buyer who needs to make an effort to close the deal. The seller (she) does not know the buyer's valuation of the good and tries to infer it from the time it takes the buyer (he) to reach the closing stage when she has to come up with a final take-it-or-leave-it offer. Buyers of different types know this and choose their efforts to close strategically.

An example of this situation is selling a startup or a firm. Closing the deal takes the buyer some time since he has to do due diligence and find financing. For illustration, Boone and Mulherin (2009) notes that a substantial portion of the deals they examined involved negotiations with a single bidder, which fits into the framework that I study. Moreover, Boone and Mulherin (2009) point to the costs associated with conducting due diligence activities as a potential explanation for the limited number of bidders.

In my model, I assume that if the buyer's enthusiasm about the startup is high, he can make more effort to accelerate the purchase, but the seller can infer his enthusiasm if the turnaround is too fast. This creates incentives for the buyer to take his time.

My objective is to characterize the dynamics of the seller's beliefs and the buyer's effort to close the deal in equilibrium. I specifically focus on one tractable type of equilibria where buyers with a high valuation exert weakly more effort than those with a low one. In these equilibria, the seller progressively gets more pessimistic since longer waiting times make it more likely that the buyer is not investing much in completing the trade, so he is more likely to be of the low type.

I show that if the buyer's cost of effort is not too high, these equilibria exhibit tipping points when the seller's posterior closing-stage beliefs jump down in the middle of the game, instantly making her more pessimistic. All of these tipping points are associated with the low-type buyer stepping up his efforts to close. Moreover, one of the tipping points is terminal: the seller's beliefs abruptly jump down to a level at which they stay thereafter. All types of buyers switch to the maximal investment, and the learning ends.

I also find that the dynamics of the buyers's effort can be non-monotonic. The players can generate a "market freeze" in the middle of the game, with the low-type buyer not investing in closing the deal and the high-type one decreasing his investment over time, even though the seller continually becomes more pessimistic, which makes closing more attractive to the buyer. This market freeze is followed by a burst of activity, with all buyer types suddenly switching to the maximum effort and accelerating the closing as much as they can.

I model the game as follows. The closing stage arrives with a certain intensity that the buyer can choose, picking from a range between two bounds. The lower bound is free, and increasing intensity entails a linear cost. The buyer's type is a distribution over his valuations of the good that will be revealed at the closing stage. These distributions can be ranked across types. The seller anticipates the level of intensity each type of buyer picks at every date and updates her beliefs accordingly as time progresses. When the closing stage happens, the seller makes a take-it-or-leave-it offer to the buyer, who then draws his valuation and decides whether to accept. I focus on equilibria, where the terms of trade gradually improve over time as the seller gets more pessimistic about the buyer getting a favorable signal about the product's value. The seller's views on the likelihood between the two buyer types change continuously as she waits, and she revises them again when the closing stage arrives (potentially with the jumps).

The seller's pessimism determines the buyer's incentives to speed up the deal closing since he expects a better deal when the seller has low expectations of his valuation. If the starting beliefs are too pessimistic, both buyer types choose the highest possible effort, and there is no learning: they are not distinguishable based on the information that the closing has not arrived. If the starting conditions are too optimistic, the buyer does not invest at all, and again, there is no learning.

For intermediate starting conditions, equilibria feature non-trivial dynamics of beliefs. One class of equilibria (that exists only when the costs of effort are relatively high) resembles the one in Kim and Pease (2017): the low type does not invest in speeding up the deals, and any updating of the seller's beliefs is only due to the high type's activity. I show that beliefs have continuous time paths and strictly decrease in time, although the high type's effort can be non-monotone. To get the intuition for this non-monotonicity, suppose that initially, the seller's beliefs are quite optimistic but the high-type buyer would still prefer closing to nothing. If a Bayesian seller's beliefs are close to one, it is hard to change them, so trying to induce pessimism by delaying the closing and mimicking the low type is not worth the time. The high type chooses the maximal level of effort. But given time, the seller's beliefs depart from one, and it becomes easier to influence them, so the high type slows down and waits for better terms of trade. Finally, when beliefs are pessimistic enough, the high type starts to exert more effort again.

After characterizing these equilibria, I depart from the parametric assumptions that either prevent learning or make effort prohibitively costly for the low type. For other parameters, concurrent investment in closing by different types induces interesting dynamics of beliefs with tipping points. Specifically, I show that a generic equilibrium with monotone beliefs features two tipping points at which beliefs abruptly jump down. Both of these tipping points are induced by upward jumps in the low type's effort. Before the first jump, he does not invest in closing at all, and after the second jump, he invests as much as possible. Moreover, the second tipping point is terminal: after that, the types choose the same effort, so additional time to closing does not tell which one of them is more likely.

There are multiple equilibria in the generic case. In some, the two tipping points collapse into one. The path of effort is, again, potentially non-monotone. Specifically, the market can freeze in the middle of the game before a burst of activity when both buyer types switch to the highest possible effort. The history of events then looks like the initial enthusiasm of the buyers and the seller's optimism are both slowly waning until, at a tipping point, the seller becomes discontinuously more pessimistic, and both buyer types accelerate. I characterize the conditions for a market freeze to happen in a tractable special case — an equilibrium with the least long-run learning. In it, the two tipping points collapse into one. The low type buyer does nothing at first and then maximizes his effort, while the high type starts out very active, then slows down, and then maximizes his effort as well.

1.1 Related Literature

A closely related paper is that by Kim and Pease (2017), who extend Mortensen (1986) by adding adverse selection and endogenous search intensity to study how the sellers allocate effort to initiate contact with the next buyer, with the option to continue the search upon negotiation breakdowns. My setup shares the basic primitives with theirs, although I modify the closing stage and obtain very different strategic dynamics. The main innovation of my paper is the active choice of effort by both buyer types, while a key assumption in Kim and Pease (2017) analysis is that a high-type seller type derives no surplus from transactions (due to assumptions about information and bargaining protocol) and consistently opts for minimal effort. This case resembles my model's equilibrium with a persistently slow low-type buyer, although the 'solicitation effect' phenomenon is absent from my model. I rely on diminishing belief as the sole incentive mechanism. My main focus is also on the concurrent activity of both types that leads to tipping points, a qualitative change in the seller's beliefs as the game progresses. I show that the low type's activity is crucial for this type of dynamics.

Guerrieri and Shimer (2014) is a seminal contribution showing that informed sellers in markets with trading frictions can sort themselves out by accepting a lower probability to trade. Giving up trading surplus can work as a signal of quality that persuades uninformed buyers to pay more.

Martel, Mirkin, and Waters (2022) study a problem of the seller who privately learns the quality of the good over time and sells to uninformed buyers. The seller chooses her selling strategy in response to both negative news about quality and exogenous selling needs. Over time, buyers update their beliefs, taking into account the seller's dynamic strategy. Martel, Mirkin, and Waters (2022) find non-monotone dynamics of prices: the decline at first, as the seller learns over time and likely becomes informed, but then rise again as informed sellers largely complete their trades and leave the market to sellers with exogenous trading needs.

In a related setup, Hwang (2018) studies trading between an uninformed buyer and a seller who gradually learns the quality of the good. The seller is initially uninformed too, but can get informed after receiving news. Buyers make offers and can switch between primarily targeting the uninformed seller and the informed one. Hwang (2018) finds a "market freeze" pattern similar to mine, characterizing a fall in trade probabilities and offered prices in the middle of the game followed by a recovery. This happens because the seller becomes sufficiently likely to be informed at some point, and the buyer starts making more conservative offers.

Daley and Green (2012) study an environment where the seller is always informed, and buyers gradually learn instead. This setup also features a period of trade collapse after which, based on the news, there is either a revival of optimism or an entrenchment of pessimism that leads the seller to concede to lower prices. Kaya and Kim (2018) describe trading between an informed seller and a sequence of imperfectly informed buyers who draw their personal signals about the quality upon arrival and observe the good's tenure on the market. In this setup, beliefs can increase or decrease over time, depending on the initial value. Relatedly, Asriyan, Fuchs, and Green (2017) study competition between two sellers with private information when quality is correlated across goods. Sellers can trade more or less actively, sometimes refusing initial offers and waiting for the next period, and buyers make their inferences based on that.

Deneckere and Liang (2006) study bargaining between the perfectly informed seller and a buyer that only learns from the bargaining process itself. They also find situations in which the market periodically comes to a halt, with the trading probabilities being low for some time, after which there is again a burst of activity. Moreno and Wooders (2016) study bargaining in decentralized markets with adverse selection. Moreno and Wooders (2010) find in a similar setup that decentralizing trade can help mitigate adverse selection, reviving trade volume in high-quality goods that is absent in the competitive benchmark.

Lauermann and Wolinsky (2017) study similar phenomena of endogenous search in a common value auction. In their model, the seller observes a private signal and decides how many bidders to invite to the auction, which exposes the participants to a participation curse. Lauermann and Wolinsky (2017) highlight that it is possible in this setting that the bids fail to aggregate information well, and there is an equilibrium bidding strategy that pools all the top bidders. Interestingly, I get a similar insight by analyzing the least informative equilibrium. Lauermann and Wolinsky (2016) is another related paper that considers a dynamic search by buyers who bear the costs of attracting offers from prospective sellers. Importantly, in their model, the seller does not observe the search history of every buyer (including calendar date), which makes the seller's problem stationary.

Chiu and Koeppl (2016) study market freezes in OTC markets where the market gets inactive due to deterioration of average asset quality. They explore market interventions that can revive trading by buying up assets of bad quality at a loss. In a related setting, Fuchs and Skrzypacz (2015) find that it can be optimal to restrict later trades, essentially prohibiting delays to battle market freezes. Camargo and Lester (2014) and Camargo, Kim, and Lester (2016) characterize market dynamics with freezes and trade-offs around government interventions.

2 Model

Consider a single seller offering a unique product, e.g., a start-up firm. At date 0, the buyer arrives at the market for the initial phase of a deal and gets a private signal about her valuation of the product. For tractability, I assume that the signal is binary and denote it as $\omega \in \{H, L\}$. I also refer to ω as a buyer's type. Let μ_0 denote the prior probability of the high type H. Time is continuous and runs over an infinite horizon $t \in [0, \infty)$. At every instant, the buyer and the market know the calendar time perfectly.

In every period t, there is some chance that the buyer returns to the seller for a closing stage of the deal. The buyer of type ω can decide how much effort $\lambda_t^\omega \in [\underline{\lambda}, \bar{\lambda}]$ to put into bringing the deal to a closing stage (if the closing has not occurred yet). I assume that effort is costly: the buyer of either type incurs a cost of $c \cdot (\lambda - \underline{\lambda})$ for some $c > 0$ when exerting effort λ but makes the transition to a closing stage more likely. In particular, if the buyer chooses effort λ during a time interval dt, then the deal moves to the closing stage during this time interval with probability λdt. The market does not observe the buyer's choice of effort and hence can only make inferences about the buyer's type from the time since the initial meeting.

Closing Stage. After the buyer arrives at the closing stage, she receives some terms of trade from the seller. For now, I model the closing stage in a reduced way. In Section 4, I provide some examples of how the price offer is formed that satisfy the basic properties of the closing stage I outline below.

Assume that if the market holds a belief $x \in (0, 1)$ of the buyer being of high type, then the buyer receives a payoff $u(\omega, x)$ if his true type is ω for $\omega \in \{H, L\}$. I assume that the terms of trade get worse from the buyer's point of view if the market is more convinced about the buyer's type being high: $u(\omega, \cdot)$ is decreasing for every $\omega \in \{H, L\}$. In addition, the higher buyer types receive a higher payoff for every given market belief

x: $u(H,x) > u(L,x), \forall x$. Furthermore, I assume that $u(\cdot,\cdot)$ satisfies *decreasing differences*: $u(H,x) - u(L,x)$ is decreasing in x. Decreasing differences imply that the high-type buyer is hurt *more* badly by the market's optimism. Intuitively, we must expect this condition to hold since the low type expects to trade less often. Many of the offers are rejected by a lower type so that any anticipated price increases do not affect him as much.

Beliefs. As the seller does not observe the effort choice, let $\tilde{\lambda}^\omega = \{\tilde{\lambda}^\omega_t\}_{t\geq 0}$ be the seller's belief about it (later, I require that the seller's belief about effort choice is correct in equilibrium).

Similarly, let $\tilde{P}^\omega_t \equiv \int_0^t e^{-\tilde{\lambda}^\omega_s} ds$ denote the seller's expectation of not hearing back from a buyer type ω by period t. Then, the seller's posterior belief at period t is

$$x_t = \frac{\mu_0 \tilde{P}^H_t \tilde{\lambda}^H_t}{\mu_0 \tilde{P}^H_t \tilde{\lambda}^H_t + (1-\mu_0)\tilde{P}^L_t \tilde{\lambda}^L_t}.$$

It is worth noting that the relevant belief for the terms of trade is the *posterior* after the seller has already met the buyer at the closing stage. It is potentially different from the *pre-closing* belief that the seller holds just before the buyer shows up for the closing stage. The latter is given by

$$\mu \tilde{\lambda}_t \equiv \frac{\mu_0 \tilde{P}^H_t}{\mu_0 \tilde{P}^H_t + (1-\mu_0)\tilde{P}^L_t}$$

Before closing, μ_t changes continuously with time. If the seller hears back from the buyer at t, she instantly updates again, and her posterior x_t can be different from μ_t due to potentially different anticipated effort levels $\tilde{\lambda}^H_t \neq \tilde{\lambda}^L_t$.

Buyer's Payoff. I assume the buyer discounts at a constant rate of ρ. Hence, if the buyer of type ω chooses the effort path $\lambda = \{\lambda^\omega_t\}$, she expects to reach the closing stage in period t with probability

$$P^\omega_t = \int_0^t e^{-\lambda^\omega_s} ds,$$

anticipating that the seller will hold belief x_t when the terms of trade are offered. By that time, she accumulates the total cost of effort that equals

$$C_t = \int_0^t e^{-\rho s} c(\lambda^\omega_t - \underline{\lambda}) ds.$$

Then, the buyer of type ω exerting efforts according to λ, while facing the seller belief path $x = \{x_t\}$ expects to receive a payoff:

$$V^\omega(\lambda^\omega, x) = \int_0^\infty P^\omega_t \lambda^\omega [e^{-\rho t} u(\omega, x_t) - C_t] dt.$$

Equilibrium. The equilibrium requires that the buyer of either type chooses the effort path optimally, given the seller's beliefs about effort are correct. In addition, I impose the following two technical restrictions on the effort by the buyer. First, I assume that the buyer's effort strategy is *Markovian*: it only depends on (the left limit of) the seller's belief x_t. Second, the buyer's strategy is *admissible*, meaning it admits at most finitely many points of discontinuities.

Definition 1. Say that $\langle \{\lambda_t^\omega\}_{t\geq 0}, x_t\rangle$ constitutes an equilibrium, if

1. $\{\lambda_t^\omega\}_{t\geq 0}$ solves the problem for buyer-type given the seller beliefs about effort $\tilde{\lambda}$,
2. the seller's beliefs are correct: $x_t = \frac{\mu_0 P_t^H \lambda_t^H}{\mu_0 P_t^H \tilde{\lambda}_t^H + (1-\mu_0) P_t^L \lambda_t^L}.\forall t$,
3. and there exists $\Lambda^\omega : [0,1] \to [\underline{\lambda}, \bar{\lambda}]$ such that $\lambda_t^\omega = \Lambda^\omega(x_{t-})$, where $x_{t-} \equiv \lim_{\delta\to 0} x_{t-\delta}$
4. $\{\lambda_t^\omega\}_{t\geq 0}$ is admissible: right continuous with left limits (RCLL) and has at most finitely many points of discontinuities.

Conditions 1 and 2 are the standard equilibrium restrictions. The assumption that the buyer's effort strategy is Markovian helps to focus on how the seller's beliefs shape the buyers' incentives to choose the effort to speed up the deal. The admissibility restriction is made purely for tractability. Importantly, it implies that at least in the long run, the effort path changes continuously for either buyer type.

3 Equilibrium Characterization

In this section, I provide a preliminary characterization of equilibria. First, I establish conditions under which equilibria with no learning, leading to constant seller beliefs, are feasible. Second, I explore situations where the two types separate in the long run: I derive conditions where low-type buyers persistently refrain from any positive investment in effort. Analyzing these scenarios sheds light on the primary driving forces of the model under simplified dynamics. Finally, I present the paper's main result: in Theorem 1, I characterize equilibria with monotone beliefs. I show that the seller gets more pessimistic about the buyer's type in any such equilibrium. Moreover, whenever the low type exerts positive effort, there is at least one jump in the equilibrium path of beliefs. I then illustrate conditions under which the market experiences a "freeze" before a boost of activity. In this case, even though conditions get more favorable as the seller gets pessimistic, the high type of buyer exerts less effort as time progresses until both types of buyer step up their efforts and try to close the deal as fast as possible.

3.1 Basic Analysis

Let V_t^ω denote the buyer's expected discounted future value given that the closing stage has not occurred by period t. Note that under any admissible path of equilibrium effort levels, the right derivative $\dot{V}_{t+}^\omega$ always exists, and the buyer's value must satisfy the following HJB equation:

$$\rho V_t^\omega = \max_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} \lambda [u(\omega, x_t) - c - V_t^\omega] + c\underline{\lambda} + \dot{V}_{t+}^\omega \tag{V-DE}$$

To get the intuition behind the HJB above, suppose that the buyer chooses λ for some short interval Δ. With probability $\lambda\Delta$, he moves to a closing stage during this interval and obtains an expected gain of $u(\omega, x_t)$. With a complementary probability, the closing stage does not arrive, and the buyer instead obtains a discounted expected value $(1 - \rho\Delta)V_{t+\Delta}^\omega$. In addition, no matter whether the closing stage occurs or not, the buyer incurs the costs of effort $c(\lambda - \underline{\lambda})\Delta$:

$$\begin{aligned} V_t^\omega &\approx \lambda\Delta u(\omega, x_t) + (1 - \lambda\Delta)(1 - \rho\Delta t)V_{t+\Delta}^\omega - c(\lambda - \underline{\lambda})\Delta \\ &\approx \Delta u(\omega, x_t) + (1 - \lambda\Delta)(1 - \rho\Delta)(V_t^\omega + \dot{V}_{t+}\Delta t) - c(\lambda - \underline{\lambda})\Delta \\ &\approx V_t^\omega - \rho V_t^\omega + \lambda\Delta\left[u(\omega, x_t) - c - V_t^\omega\right] + \Delta c\underline{\lambda} + \Delta\dot{V}_{t+}. \end{aligned}$$

That is, $u(\omega, x_t) - V_t^\omega$ captures the buyer's expected additional payoff from getting to the closing stage immediately instead of waiting. The buyer is willing to invest in effort when this expected gain of effort is higher than its marginal cost, which implies that the solution λ_t^ω must satisfy

$$\lambda_t^\omega = \begin{cases} \bar{\lambda}, & \text{if } u(\omega, x_t) - V_t^\omega > c \\ \underline{\lambda}, & \text{if } u(\omega, x_t) - V_t^\omega < c \end{cases} \tag{FOC-c}$$

In addition, whenever effort choice is interior over some time interval (τ, τ'), it must be that the buyer is exactly indifferent between waiting and closing the deal instantly, so that:

$$u(\omega, x_t) - c - V_t^\omega = 0. \tag{FOC-i}$$

Differentiating the above Equation (FOC-i) with respect to time and using Equation (V-DE), we obtain the following partial characterization of the equilibrium belief path:

$$u_x'(\omega, x_t)\dot{x}_t = \dot{V}_t^\omega = \rho V_t^\omega - c\underline{\lambda} = \rho u(\omega, x_t) - (\rho + \underline{\lambda})c \tag{x-DE}$$

It is evident that the buyer's incentives to invest in the effort are determined by the shape of $u(\omega, x_t)$. To get more structure on the incentives, I impose an additional restriction on the consumer closing-stage payoff. Say that $u(\cdot,\cdot)$ satisfies a *no-crossing condition*, meaning

$$\frac{\rho u(H,x)-(\rho+\underline{\lambda})c}{u'(H,x)} < \frac{\rho u(L,x)-(\rho+\underline{\lambda})c}{u'(L,x)}, \forall x \in [0,1].$$

This no-crossing condition concerns local incentives to invest in the effort as defined in (x-DE). In particular, suppose that both types are indifferent in period t, and the path of beliefs is such that the low type is indifferent in the neighborhood of t. Consider now incentives of the high type. The flow payoff changes by

$$u'_x(H,x_t)\dot{x}_t = u'_x(H,x_t)\frac{\rho u(L,x_t)-c(\underline{\lambda}+\rho)}{u'_x(L,x_t)} < \rho u(H,x_t)-c(\underline{\lambda}+\rho) = \dot{V}^H_{t+} \tag{1}$$

Hence, the high type must choose the lower bound $\underline{\lambda}$ in the neighborhood of t in every admissible equilibrium if at t both happen to be indifferent.

In general, the introduced setup features several different types of equilibria and potentially allows for multiple equilibria as well. In some cases, it is possible to show uniqueness. These cases are especially simple to characterize and illustrate. I next treat two such classes, one without any learning whatsoever and one without any active effort exerted by the low-type buyer.

3.2 Equilibria with No Learning

This is the simplest class of equilibria: the seller never learns about the buyer's type and consequently offers constant prices. Formally, say that an equilibrium is a *no-learning equilibrium* if posterior beliefs are never updated and stay at the prior level μ_0: $x_t \equiv \mu_0, \forall t$.

Given that the buyer's strategy is Markovian, since x_t is assumed to be constant, an equilibrium must also have constant effort levels by both buyer type $\lambda^\omega_t \equiv \lambda^\omega$. In addition, if the seller's inference about effort choice is correct, then the belief path, when continuous, satisfies:

$$\dot{x}_t = x_t(1-x_t)(\lambda^L - \lambda^H).$$

Hence, constant beliefs are only possible in equilibrium if both buyer types choose the *same* effort level: $\lambda^\omega \equiv \lambda$ for some λ. In this case, it is easy to derive a closed-form solution for each buyer's type value function:

$$V_t^\omega \equiv \frac{\lambda}{\lambda+\rho}u(\omega,\mu_0) - \frac{c(\lambda-\underline{\lambda})}{\lambda+\rho}$$

By Equation (FOC-c) and given the derived buyer-value above, type ω must choose

$$\lambda^\omega = \begin{cases} \bar{\lambda}, & \text{if } \rho u(\omega,\mu_0) > (\underline{\lambda}+\rho)c \\ \underline{\lambda}, & \text{if } \rho u(\omega,\mu_0) < (\underline{\lambda}+\rho)c \end{cases}$$

Hence, the two buyer types can only cooperate on the same effort level on the corners. Otherwise, suppose the low type exerts an interior effort. Then, the higher buyer type would always prefer to device towards $\bar{\lambda}$, as he gets a strictly higher payoff: $\rho u(H,x) > \rho(L,x) = (\underline{\lambda}+\rho)c$.

Finally, both types are willing to exert the highest effort $\bar{\lambda}$ whenever even the low type is willing to do so (symmetrically for the case of the lowest effort cooperation). I summarize these observations in Proposition 1.

Proposition 1. *An equilibrium with no learning exists if and only if one of the following holds:*

1. $\rho u(L,\mu_0) - c(\underline{\lambda}+\rho) \geq 0$
2. $\rho u(H,\mu_0) - c(\underline{\lambda}+\rho) \leq 0$

In the first case, both buyer types choose the highest effort level, and in the second case, they choose the lowest one.

3.3 Equilibria with a Persistently Slow Low Type

Suppose that equilibria with no learning do not exist: $\rho u(H,\mu_0) > c(\underline{\lambda}+\rho) > \rho u(L,\mu_0)$. Say that an equilibrium is *an equilibrium with a persistently slow low type* if the low type never invests into effort $\lambda_t^L = \underline{\lambda}$.

To analyze such equilibria, I first make the following observation about potential jumps in the seller posterior beliefs: the jumps in posterior beliefs must occur simultaneously with the jumps in the effort by a low-type buyer.

Lemma 1. *If there is a jump in the equilibrium path of posterior beliefs x_t, it must coincide with a jump in the low type's effort λ_t^L.*

Lemma 1 thus states that the jumps in posterior beliefs must occur simultaneously with the jumps in the effort by a low-type buyer. I now go over the argument for Lemma 1.

If the seller holds correct beliefs about the effort, then any jumps in the posterior beliefs must coincide with a jump in the effort for at least some buyer type. I now explain why it is not possible that the high type makes a jump in her effort alone.

Suppose that at period t, there is an upward jump in the effort by the high type (but the low type has no jumps in his effort choice). Then, if the seller holds correct beliefs, the seller beliefs must also jump upwards at this period t. But then the high-type buyer must exert the highest effort possible just before the jump, at $t-$ to reach the closing stage before the terms of trade get much worse. This makes an upward jump by a high type impossible.

Formally, since the high-type exerts at least an interior effort at t, by Equation (FOC-i): $\rho u(H, x_t) - V_t^H \geq c$. By Equation (V-DE), the value function of either buyer type admits no jump, while $u(H, x_t) \ll u(H, x_{t-})$ due to an adversarial jump in the seller's beliefs. Hence, if the buyer is willing to exert some effort t, he must exert the highest effort at $t-$ due to Equation (FOC-c).

By a symmetric argument, we can rule out the case when the high type makes a downward jump in her effort while the low type's effort changes continuously.

Lemma 1 then implies that any equilibrium with persistently slow low type must have a continuous path of beliefs x_t (as the low type makes no jumps in effort by assumption), which in turn means that there can be no jumps in the effort choice by a high type λ_t^H.

Moreover, it is easy to derive that the prior belief μ_t satisfies the following ODE whenever continuous:

$$\dot{\mu}_t = \mu_t(1 - \mu_t)(\lambda_t^L - \lambda_t^H)$$

so, in any equilibrium with a persistently slow low type, μ_t decreases with time.

When both buyer types exert the same effort, the prior and the posterior seller beliefs coincide, as meeting a buyer for the closing stage brings no additional information to the seller. But then, if the high type is not willing to choose $\underline{\lambda}$ at $t = 0$,[1] then it is not possible that he exerts a low effort at any later date. At any equilibrium, the seller would have strictly more pessimistic beliefs at any date where $\lambda_t^H = \lambda_t^L = \underline{\lambda}$ compared to $t = 0$, which would give incentives to the high type to speed up the deal.

In this case, if the path of high type's effort is admissible, we should be able to find a finite collection of thresholds $\{\tau_1, \ldots, \tau_N\}$, such that for every $t \in (\tau_i, \tau_{i+1})$, either the upper

[1] Which is guaranteed by the parametric assumption $\rho u(H, \mu_0) > c(\underline{\lambda} + \rho)$.

bound is chosen and the seller beliefs evolve according to:

$$\dot{x}_t = x_t(1-x_t)(\underline{\lambda} - \bar{\lambda})$$

or the high type is kept indifferent on an interval (τ_i, τ_{i+1}):

$$\dot{x}_t = x_t(1-x_t)\left(\underline{\lambda} - \lambda_t^H + \frac{\dot{\lambda}_t^H}{\lambda_t^H}\right) = \frac{\rho u(H, x_t) - c(\underline{\lambda} + \rho)}{u'_x(H, x_t)}$$

We can now characterize the belief path in any equilibrium with a persistently slow type. Proposition 2 established that in any such equilibrium, the seller gets monotonically pessimistic about the buyer's type with time. As mentioned in the Introduction, Proposition 2 stands in contrast with Kim and Pease (2017), as the solicitation curse can never be strong enough in the equilibrium to create any non-monotonicity.

Proposition 2. *Suppose that $u(H, \mu_0) > c(\underline{\lambda} + \rho)$. In any equilibrium with a persistently slow type, posterior belief x_t is continuous and strictly decreasing.*

Proof. The continuity follows from Lemma 1. To prove the second part, suppose the contrary: x_t increases over some interval (τ_i, τ_{i+1}). First, note that $\tau_{i+1} < \infty$: based on the law of motion for the seller's posterior beliefs, x_t can only increase, even weakly, when the high-type buyer chooses an interior level of effort that is increasing sufficiently fast. Keeping $\dot{\lambda}_t^H$ sufficiently high forever is not sustainable, as eventually the upper bound on λ_t^H will bind. Hence, either our premise is wrong, or there is some point where the high type must switch to choosing $\bar{\lambda}$. Then, at τ_{i+1}, the buyer switches to $\bar{\lambda}$, and the seller's posterior beliefs decrease. However, in this case, λ_t^H is not Markovian, as in the proximity of the switching point τ_{i+1}, the buyer must choose different effort levels for the same posterior beliefs x_t (due to continuity of the belief-path). $\square$

Given that x_t is strictly decreasing, we can make a change of variables and find $\Lambda^H(x_t) = \lambda_t^H$, in addition whenever $\Lambda^H(x)$ is interior, it must satisfy the following ODE:

$$\frac{\partial \Lambda^H(x)}{\partial x} = \frac{\dot{\lambda}_t^H}{\dot{x}_t} = \Lambda^H(x)\left[\frac{1}{x(1-x)} + (\Lambda^H(x) - \underline{\lambda})\frac{u'_x(H,x)}{\rho u(H,x) - c(\underline{\lambda}+\rho)}\right] \qquad \text{(H-}\lambda\text{)}$$

As x_t converges to 0, the value function converges to $V_\infty = \frac{\bar{\lambda}}{\bar{\lambda}+\rho}u(H,0) - c\frac{\bar{\lambda}-\underline{\lambda}}{\bar{\lambda}+\rho} < u(H,0) - c$. Hence, it must be that the high-type seller eventually chooses the highest possible effort $\bar{\lambda}$. Since λ_t^H is right continuous, it must be that $\Lambda^H(0) = \bar{\lambda}$. Iterating backward the value function and $\Lambda^H(x)$, there is a unique way to make the buyer's choice consistent with

(FOC-c) while preserving continuity in λ_t^H. I use the same backward construction below when characterizing a different type of equilibrium and verify that it converges if the buyer's utility function is sufficiently well-behaved.

Definition 2. Say that $u(\omega, x)$ is regular if its first derivative is bounded, and the function $\rho u(H,x) - c(\underline{\lambda} + \rho) - x(1-x)(\bar{\lambda} - \underline{\lambda})u_x'(H,x)$ is analytic.

Finally, it remains to verify that the low type is willing to choose the lowest possible level of effort forever. First, note that x gets almost constant as it approaches 0. From the previous section, we know that under constant x, the low type is not willing to invest in effort if and only if $\rho u(L,0) - c(\underline{\lambda} + \rho) \le 0$. Lemma 9 verifies that this is the tightest incentive compatibility constraint on the low type, meaning that if he does not wish to invest at $x = 0$, he will not do so at all earlier dates given the equilibrium belief path.

Proposition 3. *Suppose that $u(H,x)$ is regular and $\rho u(H,\mu_0) > c(\underline{\lambda} + \rho)$. An equilibrium with a persistently low type exists and is unique if and only if $\rho u(L,0) - c(\underline{\lambda} + \rho) \le 0$.*

Proposition 3 highlights the following observation. Efforts by the high type force equilibrium beliefs to get more and more pessimistic. And since the high type never stops exerting effort, the seller becomes convinced that it must be the low type if closing happens late. But this, in turn, incentivizes the low type to induce effort. Hence, the low type can only be prevented from investing in effort when the costs are prohibitively high, even under the extreme seller's beliefs.

Another property of this equilibrium is that the high type's effort can be non-monotone in time, decreasing at first and increasing again later. This is possible if his utility is high enough, even when the seller is sure that the valuation is high.

Proposition 4. *Suppose that μ_0 is sufficiently high and $\rho u(H,1) - c(\underline{\lambda} + \rho) > 0$. Then, either $\lambda_t^H = \bar{\lambda}$, or the effort is non-monotone.*

The intuition for the potential non-monotone dynamics of effort from this proposition is as follows. If the seller's belief is sufficiently close to one at first, persuading her to delay the deal is hard in equilibrium. Indeed, it is easy to notice from (H-λ) that for high enough x, $\Lambda^H(x)$ increases in x as $1/[x(1-x)]$ diverges, meaning that the difference in intensities between the types should be very large to move beliefs. Mimicking the low type by delaying the closing is not worth the time because the buyer is impatient, and the beliefs move slowly near 1. This changes when beliefs move further away from one, and the high type must slow down in equilibrium to wait until the buyer accumulates pessimism and is ready for better terms. Finally, when the seller's beliefs are pessimistic enough, it becomes harder to persuade her again, and the high type increases his efforts.

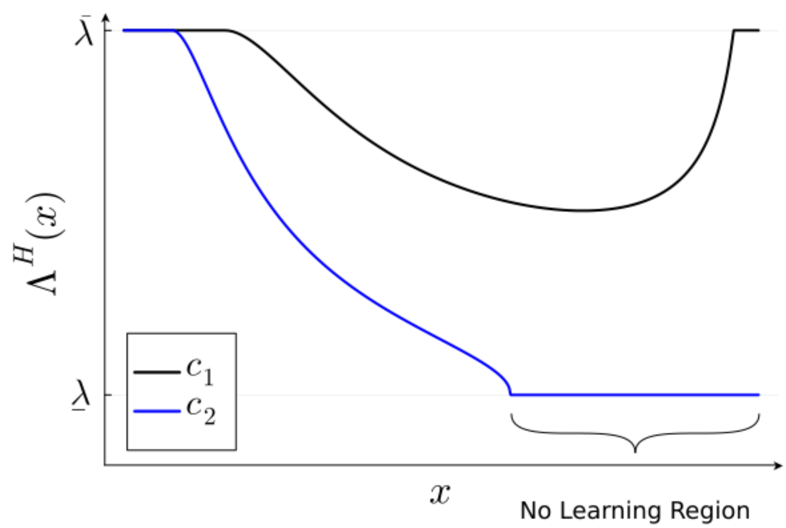

Figure 1: Effort Strategy by a High Type in an Equilibrium with a Persistently Slow Type

Note: computed for $G^H(\varepsilon) = \varepsilon^2$, $c_1 < c_2$. With higher costs c_2, if the prior belief is in the no-learning region, both buyer types choose $\underline{\lambda}$ forever, and there is no learning. With low costs c_1, $\rho u(H, 1) > c(\underline{\lambda} + \rho)$, so that effort choice exhibits non-monotonicity if the prior is close to 1.

3.4 Equilibria with an Active Low Type

I now move to analyze the most interesting case, where both buyer types actively exert effort to reach the closing stage sooner. In particular, I assume that the investment costs are such that none of the previously considered equilibria exist: $\rho u(L, 0) > c(\underline{\lambda} + \rho) > \rho u(L, \mu_0)$ and $\rho u(H, \mu_0) > c(\underline{\lambda} + \rho)$.

I first verify whether it is possible to obtain an equilibrium where the posterior beliefs x_t are continuous. Lemma 2 implies that no such equilibria exist under the imposed parametric assumptions. Indeed, note that it is not possible that the belief path x_t is continuous and non-monotone, as otherwise, the buyer's strategy is not Markovian. But then, it must be that the belief has a long-run limit. Lemma 2 now shows that x_t cannot reach this limit continuously, meaning that there must be at least one jump.

Lemma 2. *Suppose no equilibria with no learning or a persistently slow low type exist. Then, in any equilibrium there exists a limit value of posterior beliefs x_∞ and x_t reaches its long-run limit with a jump: that is, there exists τ such that $x_t = x_\tau$ $\forall t \geq \tau$, and $\lim_{\delta \to 0} x_{t-\delta} \neq x_\tau$.*

Proof. By admissibility, a point τ exists, such that x_t is continuous for $t \geq \tau$. Find the lowest such τ, after which x_t is continuous. If x_t is constant on $[\tau, \infty)$, we are done. If not, then it must be monotone on $[\tau, \infty)$, as otherwise, the buyer's strategy is not Markovian (see Lemma 7 for details). Given that x_t is monotone and bounded, its long-run limit must exist.

If both buyer types strictly prefer the same boundary at x_∞, then the preference is preserved in the proximity of x_∞, and the beliefs must have stopped evolving sooner. In particular, given parametric assumptions on c, both types choose $\bar{\lambda}$ in the proximity of 0, hence $x_\infty > 0$. Now assume that $x_\infty = 1$, then the low type chooses $\underline{\lambda}$ in the proximity of x_∞. As x_t reaches its upper bound, it must be increasing, which is only possible when the high type chooses an interior level of effort. To make x_t increasing near 1, $\frac{\dot{\lambda}_t^H}{\lambda_t^H} \geq \lambda_t^H - \underline{\lambda}$ which is exploding. Then, $x_\infty \in (0,1)$ and it is not possible that the two types strictly prefer two opposing boundaries at the limit (as otherwise x_t escapes x_∞).

Hence, at least one buyer type must be indifferent at the limit. Suppose type ω is indifferent at the limit. Then, the other type $-\omega$ has a strict preference at x_∞ preserved in its proximity given the continuity assumption on x_t given the observation in Section 3.2.

Moreover, if x_t is decreasing (increasing) as it reaches x_∞, in the proximity of x_∞, type ω strictly prefers $\underline{\lambda}$ ($\bar{\lambda}$) [2]. Hence, if x_t is decreasing, then the high type cannot be indifferent at x_∞, as otherwise, x_t would have stopped evolving sooner. Symmetrically, it cannot be that the low type is indifferent when x_t is increasing as it reaches x_∞. In addition, if x_t is increasing and the high type is indifferent at x_∞, then $\dot{x}_t = x_t(1-x_t)(\bar{\lambda}-\underline{\lambda})$ in the proximity of x_∞ and we get immediate contradiction.

Suppose that x_t is decreasing and the low type is indifferent at x_∞. Then, in the proximity of x_∞, $\dot{x}_t = x_t(1-x_t)(\bar{\lambda}-\underline{\lambda})$. But then, x_t reaches x_∞ in finite time [3]. After x_∞ is reached, the posterior belief must remain constant. If there are no jumps, the only way to preserve x_∞ is to have $\frac{\dot{\lambda}_t^L}{\lambda_t^L} = \lambda_t^L - \bar{\lambda}$. But at the point when the limit is reached $\lambda_t^L = \underline{\lambda}$, hence we cannot have that $\dot{\lambda}_t^L < 0$. Contradiction. □

However, as shown in Lemma 1, this scenario does not inherently present any obstacles to equilibrium existence. As long as the constraints allow the low type's effort to jump as much as needed to be consistent with a given jump in x_t, such scenarios are at least feasible from the seller's Bayesian updating perspective.

Dealing with equilibria featuring multiple jumps can be complex. Therefore, I focus on well-disciplined ones. Specifically, I analyze equilibria where the direction of the seller's learning remains constant. Proposition 1 confirms that in such cases, the only possibility is that the seller becomes progressively more pessimistic about the buyer's type. To that end, it is sufficient to establish that no equilibrium admits upward terminal jumps. Indeed, by Lemma 2 any monotone belief path must reach its limit with a jump. If upward terminal jumps are not possible, then there is no way that an equilibrium belief path x_t is monotone.

[2] Otherwise, we would get a contradiction with (x-DE).

[3] Solving the differential equation for $x_{t+\tau} = x_t / \left(x_t + (1-x_t)e^{(\bar{\lambda}-\underline{\lambda})\tau} \right) \to 0$ for every $x_t > 0$, hence if x_∞ is interior, it is reached in finite time.

Lemma 3. *No equilibrium admits upward terminal jumps.*

I delegate the full proof to Section D. The proof relies on the decreasing differences property of the expected utility functions. In particular, this property implies that if the low type is to make a downward jump towards $\underline{\lambda}$, it must be that the high type jumps to $\underline{\lambda}$ all the way from $\bar{\lambda}$. There is no way to compensate for this jump by a high type, making an upward jump inconsistent with equilibrium behavior.

We are now ready to formulate the main result characterizing the equilibria with monotone beliefs.

Theorem 1. *Suppose that $u(\omega, \cdot)$ is regular for every buyer type. An equilibrium with monotone beliefs exists. Moreover, in any equilibrium, if x_t is monotone, then it is decreasing. If $\bar{\lambda}$ is sufficiently high, then for every such equilibrium, there exist two thresholds $\tau_1 \leq \tau_2$ such that*

1. *The low type does not invest until τ_1, $\lambda_t^L = \underline{\lambda}, \forall t < \tau_1$;*
2. *The high type chooses the highest effort after τ_1: $\lambda_t^H = \bar{\lambda}, \forall t \geq \tau_1$;*
3. *Beliefs jump downward at both τ_1 and τ_2, moreover the second jump is a terminal jump: $x_t = x_{\tau_2}, \forall t \geq \tau_2$;*

Proof. I prove existence by construction in the next section. To get the properties of all monotone equilibria, note that by Lemma 2, x_t must reach its limit with a terminal jump. By Lemma 3, upward terminal jumps are impossible. Then, an equilibrium belief path is decreasing whenever it is monotone. Hence, there exists τ_2, such that x_t is remains constant after τ_2. There are just two options for how a downward (to preserve monotonicity) jump is possible: either the low type jumps from an interior effort or from $\bar{\lambda}$. Supporting Lemmas 8 and 9 in section D establish that the no-crossing condition on $u(\cdot, \cdot)$ guarantees that if the low type is indifferent on an open interval before the jump (τ, τ_2), then the high type prefers $\bar{\lambda}$ on that interval. Or if, instead, the low type prefers $\underline{\lambda}$ at τ_2, then the low must choose $\underline{\lambda}$ on any open interval where x_t is continuous before the jump (provided that $\bar{\lambda}$ is high enough). In the latter case, no jumps are possible since the low type cannot jump upwards towards $\underline{\lambda}$: $\tau_1 = \tau_2$.

In the former case, the low type can make a jump towards an interior effort level, but only from $\underline{\lambda}$ (it is not possible to keep the buyer indifferent with the jump, since V_t^ω is continuous). Hence, it is only possible that there is one more jump at τ_1. □

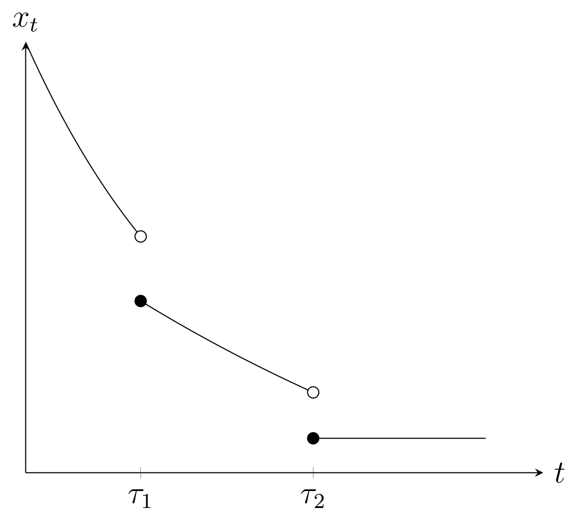

Figure 2: Equilibrium Belief Path with Active Low Type

3.5 Least Informative Equilibrium

I now focus on a special equilibrium with monotone beliefs. Specifically, I explore the equilibrium with the least long-run learning. Since in every such equilibrium, the seller must get more pessimistic with time as established in Theorem 1, in the one with the least learning, the long-run posterior beliefs are the highest. It is the simplest type of equilibrium to describe since, as I explain below, it only features one jump in beliefs, after which there is no more learning, and both buyer types work as actively as possible to close the deal. Before the jump, however, the market can even slow down over time, with the low type not making any effort and the high type investing less and less despite the seller becoming more pessimistic.

Definition 3. Say that an equilibrium with monotone beliefs is the least informative in the long run if every other such equilibrium has a lower long-run posterior belief x_∞.

Suppose that a no-learning equilibrium does not exist. Then, either the low type is persistently slow, or the equilibrium belief path strictly decreases and jumps to its limit value. The downward jumps are only possible if the low type is willing to step up effort at some point, which implies that he is willing to exert effort when x_t is at its long-run limit. By the same reasoning as in Section 3.2, it must be that $\rho u(L, x_\infty) - c(\underline{\lambda} + \rho) \geq 0$.

Hence, the highest belief that can be supported in equilibrium is where this inequality binds [4]:

$$\rho u(L, \tilde{x}) - c(\underline{\lambda} + \rho) = 0$$

[4] For the construction of an equilibrium with a persistently slow low type, we take $\tilde{x} = 0$.

In general, it is hard to rank all the equilibria based on their surpluses for either side of the market since they depend on the whole dynamics of the system. However, it is easily verified that the suggested equilibrium has the highest long-run stationary profit for the seller and the lowest long-run expected surplus for the buyers.

By (x-DE), it follows that it cannot be that the low type chooses an interior level of effort before the terminal jump and x_t is decreasing. Hence, in this equilibrium, the second interval is empty ($\tau_1 = \tau_2 \equiv \tau$), and the low type must jump from $\underline{\lambda}$ to $\bar{\lambda}$. In addition, by the proof of Lemma 9, the low type never exerts effort at any earlier date.

It remains to construct the equilibrium belief path and the effort path for the high type. Just before the jump, the high type must be willing to exert effort so that x_t continues decreasing. Hence, it must be that

$$u(H, x_{\tau-}) - c \geq V_\tau^H = \frac{\bar{\lambda}}{\bar{\lambda} + \rho} u(H, \tilde{x}) - \frac{c(\bar{\lambda} - \underline{\lambda})}{\bar{\lambda} + \rho}$$

This condition bounds the magnitude of the jump that can happen as the low type starts exerting effort at τ. Depending on the value of $\tilde{x}$ and the shape of $u(H, \cdot)$, either the high type chooses some interior effort just before the jump, or his utility function is sufficiently high at $\bar{\lambda}\tilde{x}/(\bar{\lambda}\tilde{x} + (1-\tilde{x})\underline{\lambda})$ so that he is willing to invest up to $\bar{\lambda}$ even with the most dramatic jump in the posterior belief. Either way, we get an initial condition from which we can iterate backward and finish the construction of the higher type's effort. I provide the exact construction in Section E.

Interestingly, it is possible that in this equilibrium, the high type exerts less effort with time, even though the flow payoff increases due to more favorable beliefs. The market slows down before the jump, after which it gets very active, with both types of buyers trying to close the deal as soon as possible. Proposition 5 summarizes some sufficient conditions for when the market should be expected to slow down in the equilibrium we have just constructed.

Proposition 5. *Suppose that $\mu_0 > \tilde{x} > 0$. The market slows down (λ_t^H is decreasing) in the neighborhood of the jump in the least informative equilibrium whenever one of the following is true:*

1. *The two types are sufficiently similar at $\tilde{x}$: $u(H, \tilde{x}) - u(L, \tilde{x}) \to 0$*
2. *If the upper bound of effort choice $\bar{\lambda}$ is sufficiently large: $\bar{\lambda} \to \infty$*

The proof is delegated to Section D.

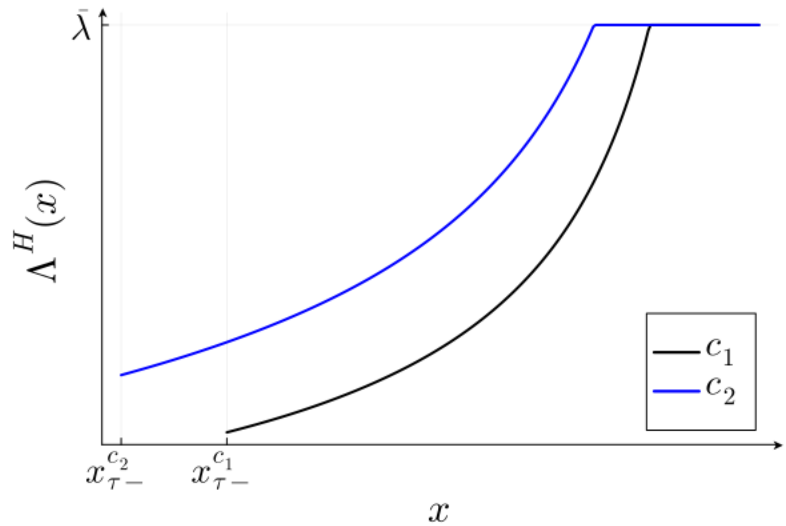

Figure 3: High Type's Effort Strategy

Note: high type buyer's effort choice as a function of the seller's posterior belief x. The optimal strategy is computed for $G^H(\varepsilon) = \varepsilon^2$ and $G^L(\varepsilon) = \varepsilon$, with $c_1 < c_2$. In equilibrium, the posterior belief is decreasing, hence the plot suggests that the high type's effort is decreasing with time before the terminal jump.

Figure 3 suggests the costs of effort may also have some unexpected consequences for the choice effort: with higher costs, the high-type buyer exerts more effort around the jump point, and the market slow-down is less dramatic. It would be interesting to explore what conditions on the $u(\cdot,\cdot)$ entail such dynamics in future work. Exploring the dynamics of effort in other equilibria also seems to be an interesting direction for future research but remains outside of the project in its current form.

4 Discussion: Closing Stage

Note that the main results only rely on the properties of the buyer's expected utility functions. In this section, I provide some examples of the closing stage leading to the buyer's utility with these properties, examining different information structures and bargaining protocols.

For both of the following two examples, I assume that the buyer's initial signal about the product value is imperfect. For interpretation, the financial intermediary may get a poor initial signal about the start-up quality but may hope to get favorable financing options that would make the deal worth it.

In particular, I assume that once the buyer arrives for the closing stage, she draws a deal-specific shockε, which is drawn according to some known cdf G^ω that depends on the buyer's initial signal ω. I assume that G^ω admits a strictly positive pdf g^ω over the common support $[\underline{\varepsilon}, \bar{\varepsilon}]$ In particular, I assume that the high signal carries favorable news about the deal-specific shock in terms of monotone likelihood ratio property (MLRP) order.

Definition 4. Say that G^H dominates G^L in MLRP order ($G^H \succ_{MLRP} G^L$), if the ratio of pdfs $g^H(\varepsilon)/g^L(\varepsilon)$ is strictly increasing in ε.

4.1 Take-it-or-leave-it Offer

For this specification of the model, assume that once the buyer gets to a closing stage, he gets a take-it-or-leave-it offer from a seller. Upon getting the offer from a seller, the buyer observes a deal-specific shock ε and decides whether to accept the price. If the offer is accepted, the seller gets a payoff of p, and the buyer receives $\varepsilon - p$. If the price offer is rejected, both sides get 0.

I assume that neither of the parties has any commitment power for their decisions. In particular, I require the buyer to accept any price offer below their deal-specific shock ε: they cannot make threats in advance of the closing. Given the buyer's acceptance decision, the seller chooses her price offer in the closing stage based on her posterior beliefs about the buyer's type x_t if closing happens at t. The seller's expected payoff from offering a price p when her beliefs are x_t is

$$p\left[x_t(1 - G^H(p)) + (1 - x_t)(1 - G^L(p)\right]$$

This has a unique maximizer for every x_t as long as the distribution of deal-specific shocks (for both types) has an increasing hazard rate $\frac{g^\omega(x)}{1-G^\omega(x)}$.

In addition, it can be shown that the optimal price choice is increasing in x_t as long as the high signal H carries more favorable news about the deal-specific shock ε. This is very intuitive: the more convinced the seller is that the buyer's initial signal is high, the more optimistic she is about the buyer's final evaluation of her product, and higher prices become more profitable in expectation.

Lemma 4. *Suppose that $G^H \succ_{MLRP} G^L$ and both g^H and g^L are increasing. Then, there exists a unique $\mathbf{p}(x_t)$ that maximizes the seller's expected profit for posterior beliefs x_t. In addition, $\mathbf{p}(\cdot)$ is an increasing function.*

Proof. Delegated to Section A. □

Then, given the belief path $\{x_t\}$ of the seller, let $u : \{H, L\} \times [0, 1] \to \mathbb{R}$ define the buyer's interim expected payoff if they reach the closing stage in period t:

$$u(\omega, x_t) \equiv \int_{\mathbf{p}(x_t)}^{\bar{\varepsilon}} (\varepsilon - \mathbf{p}(x_t)) dG^{\omega}(\varepsilon)$$

I next show that the monotone likelihood ratio property implies that the buyer types can be ranked by their closing utility, that they benefit from the seller's pessimism, and that both the decreasing differences and no-crossing conditions are satisfied.

Lemma 5. *Suppose that $G^H \succ_{MLRP} G^L$ and both $g^H(\cdot)$ and $g^L(\cdot)$ are increasing. Then, $\{u(\omega, x)\}_{\omega \in \{H,L\}}$ has the following properties:*

1. $u(H, x) > u(L, x)$*, for every* $x \in [0, 1]$
2. $u(\omega, \cdot)$ *is decreasing for every* $\omega \in \{H, L\}$
3. *decreasing differences*
4. *no-crossing whenever* $\rho u(H, 0) - c(\rho + \underline{\lambda}) > 0$

Proof. Delegated to Section B. □

4.2 Adverse Selection and Random Proposals

Suppose that the seller's expected value for the product is now given by v_ω, where $v_H > v_L$ — that is, the buyer's value for the product is positively correlated with the seller's. As before, the seller does not observe the realization of the buyer's signal and has to make inferences based only on the calendar date. Upon the arrival of the deal closing stage, the two parties get a random price $p \in [0, \bar{p}]$ distributed according to some cdf F with a positive pdf f. Once both parties observe the price, they each decide whether the price is acceptable to them. If both parties agree, the transaction occurs, or the deal breaks down without renegotiation. This bargaining protocol is borrowed from Lauermann and Wolinsky (2016) to abstract away from the difficulties of modeling bargaining with asymmetric information.

The seller's behavior in the closing stage is then as follows. For every drawn price p, and a posterior belief x, she compares the expected payoff of accepting the price

$$p \cdot \left[x(1 - G^H(p)) + (1 - x)(1 - G^L(p))\right] + x \cdot v_H \cdot G^H(p) + (1 - x) \cdot v_L \cdot G^L(p)$$

the expected payoff of keeping the product: $xv^H + (1 - x)v^L$. In particular, given the belief x about the buyer's signal, the seller accepts price p when: In the framework, provided that the adverse selection effect is not too severe, it is optimal for the seller to adopt a threshold strategy: that is, for every posterior belief x_t, there exists $\mathbf{p}(x_t)$, such that the seller accepts all prices above $\mathbf{p}(x_t)$ and rejects all the prices that are below (see Section F for the proof).

Lemma 6. *Suppose that $G^H \succeq_{MLRP} G^L$, and suppose that adverse selection is not too severe: $1+(v_H-v_L)\left(\frac{g^H(\varepsilon)}{1-G^H(\varepsilon)} - \frac{g^L(\varepsilon)}{1-G^L(\varepsilon)}\right) \geq 0$ over the support of ε. Then the seller's optimal strategy is a threshold rule: the seller accepts a price if it is higher than $\mathbf{p}(x)$. In addition, $\mathbf{p}(x)$ is increasing in x.*

Given the seller's strategy above, the buyer's expected surplus from closing a deal when the seller's posterior belief is x is then:

$$u(\omega, x) = \int_{\mathbf{p}(x)}^{\bar{\varepsilon}} \int_{\mathbf{p}(x)}^{\varepsilon} \varepsilon - p dF(p) dG^{\omega}(\varepsilon)$$

It can be shown that under the premise of the above lemma Lemma 5 holds for the utility functions as above (see Section F for the proof).

5 Conclusion

This paper develops a theory of frictional trade with adverse selection and endogenous effort that accelerates closing the deal. I show that dynamics of concurrent effort by different types of buyer lead to tipping points when posterior beliefs of the seller abruptly jump down and trading accelerates. I also show how these tipping points can be preceded by market freezes that endogenously resolve themselves with both types of buyer stepping up their efforts.

My results leave several venues for future work. One possibility to extend the model is to explore simultaneous search and competition between buyers, adding that to the implicit competition between different types of the same buyer. Letting the seller obtain information on her own, adding that to the inference she makes from the time to closing, could meaningfully change the dynamics of effort too.

It is also beyond the scope of this paper to show how different equilibria are ranked by different buyer types, by the seller, and by a utilitarian social planner. Another venue for future work is to investigate whether closing the deal takes inefficiently long and how subsidizing or discouraging effort might improve on the competitive equilibrium.

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6 Appendices

A Proofs for Section 3

Proof of Lemma 4. First, I establish the seller's problem

$$\max_p \quad p\left[x(1-G^H(p))+(1-x)(1-G^L(p))\right] \tag{2}$$

admits a unique maximizer as long as both types' distributions have increasing hazard rates and $G^H \succ_{MLRP} G^L$.

$$\frac{\partial}{\partial p}: x(1-G^H(p))+(1-x)(1-G^L(p))-p\left(xg^H(p)+(1-x)g^L(p)\right)$$

First note that if $p > \frac{(1-G^H(p))}{g^H(p)}$, then seller's expected profit is decreasing in p. Indeed, since $G^H \succ_{MLRP} G^L$, then $\frac{(1-G^H(p))}{g^H(p)} \geq \frac{1-G^L(p)}{g^L(p)}$ and $p > \frac{(1-G^L(p))}{g^L(p)}$. Summing up the two inequalities with weights x and $(1-x)$ delivers the result. Symmetrically, the seller's profit is increasing in p if $p < \frac{1-G^L(p)}{g^L(p)}$. By assumption, both pdfs are strictly positive over the support, and inverse hazard rates are decreasing. Hence, an intersection of the first derivative with 0 exists. Clearly, the problem is concave provided pdfs are increasing for both buyer types:

$$\frac{\partial^2}{\partial p^2}: -2\left(xg^H(p)+(1-x)g^L(p)\right)-p\left(x{g^H}'(p)+(1-x){g^L}'(p)\right)<0$$

Now, I verify that the optimal choice of price increases with x. The sign of $\mathbf{p}'(x)$ is determined by $\frac{\partial^2}{\partial p \partial x}$ evaluated at $\mathbf{p}(x)$:

$$\begin{aligned} G^L(\mathbf{p}(x))-G^H(\mathbf{p}(x)-\mathbf{p}(x)g^H(\mathbf{p}(x))+\mathbf{p}(x)g^L(\mathbf{p}(x)) \geq \\ G^L(\mathbf{p}(x))-G^H(\mathbf{p}(x)-\mathbf{p}(x)\frac{g^H(\mathbf{p}(x))}{1-G^H(\mathbf{p}(x))}\left(G^L(\mathbf{p}(x))-G^H(\mathbf{p}(x))\right) \\ =\left(G^L(\mathbf{p}(x))-G^H(\mathbf{p}(x))\right)\left(1-\mathbf{p}(x)\frac{g^H(\mathbf{p}(x))}{1-G^H(\mathbf{p}(x))}\right) \geq 0 \end{aligned}$$

Where the first inequality used $\frac{g^H(\varepsilon)}{1-G^H(\varepsilon)} \geq \frac{g^L(\varepsilon)}{1-G^L(\varepsilon)}$, and the second inequality uses earlier observation that at the optimum, $\mathbf{p}(x) \leq \frac{1-G^H(\mathbf{p}(x))}{g^H(\mathbf{p}(x))}$.

□

B Proofs for Section 3.1

Proof of Lemma 5. (1) follows directly from $G^H \succ_{MLRP} G^L$. Note that $u'_x(\omega, x) = -\mathbf{p}'(x)(1-G^\omega(x))$ and $\mathbf{p}(x) < \bar{\varepsilon}$. By Lemma 4, $\mathbf{p}'(x) > 0$, so that $u'_x(\omega, x) < 0$. In addition, $G^H \succ_{MLRP} G^L$ implies that $u'_x(H, x) < u'_x(L, x)$ and we get (3). Finally, consider (4).

Case 1: $\rho u(L, 0) - c(\rho + \underline{\lambda}) < 0 < \rho u(H, 1) - c(\rho + \underline{\lambda})$. In this case, the inequality is trivially satisfied as the RHS is positive while the RHS is negative.

Case 2: suppose that $\rho u(H, \hat{x}) - c(\rho + \underline{\lambda}) = 0$ for some $\hat{x} \in (0, 1)$. Define $\tilde{x}$ to be the posterior belief where $\rho u(L, \tilde{x}) - c(\rho + \underline{\lambda}) = 0$ (if there is no such x, let $\tilde{x} = 0$). By (1), $\tilde{x} < \hat{x}$. Note that the inequality is satisfied on $[\tilde{x}, \hat{x}]$ for the same reason as in Case 1. For $x < \tilde{x}$, the desired inequality is satisfied if and only if:

$$\frac{\rho u(H, x) - (\rho + \underline{\lambda})c}{\rho u(L, x) - (\rho + \underline{\lambda})c} > \frac{u'(H, x)}{u'(L, x)}$$

Suppose there exists some $x^1 < \tilde{x}$, where the expression above holds with equality. Consider a derivative of the LHS at x^1. By the premise, the derivative of the LHS at x^1 is 0. And the derivative of the RHS is:

$$\frac{u'(H, x)}{u'(L, x)} = \left(\frac{1 - G^H(\mathbf{p}(x))}{1 - G^L(\mathbf{p}(x))}\right)'_x = \\ \mathbf{p}'(x)\frac{-g^H(\mathbf{p}(x))(1 - G^L(\mathbf{p}(x))) + g^L(\mathbf{p}(x))(1 - G^H(\mathbf{p}(x)))}{(1 - G^L(\mathbf{p}(x)))^2} > 0$$

But then the premise must be wrong since the inequality holds at $\tilde{x}$, and the RHS is increasing in x. Hence, if it holds at a higher x, it must hold at a lower x (given that it is locally constant by the premise). Hence, the inequality holds for $[0, \hat{x}]$. It remains to very that it is also satisfied on $[\hat{x}, 1]$. In this case, the desired inequality holds whenever:

$$\frac{\rho u(H, x) - (\rho + \underline{\lambda})c}{\rho u(L, x) - (\rho + \underline{\lambda})c} < \frac{u'(H, x)}{u'(L, x)}$$

The inequality is satisfied at $\hat{x}$ (RHS is positive, while LHS is 0). Hence, there can be no point $x^1 > \hat{x}$, where the LHS is flat and simultaneously attains an increasing RHS. □

C Proofs for Section 3.3

Lemma 7. *In every equilibrium, it is not possible that x_t is continuous on an interval (τ, τ') and is increasing on (τ_0, τ_1) but is decreasing on (τ_1, τ_2) for $(\tau_0, \tau_2) \subseteq (\tau, \tau')$.*

Proof. Suppose otherwise. Then, in the neighborhood of τ_1, both buyer types must choose λ_t^ω as their strategies are assumed to be Markovian. Then, the only possibility for the premise to be true is if both to the left and to the right of τ_1, at least one consumer type is indifferent. Due to (x-DE), such a change in the sign of $\dot{x}_t$ is only possible when the low type is indifferent before τ_1, and the high type — after τ_1 with:

$$\rho u(L, x_{\tau_1}) - c(\underline{\lambda} + \rho) < 0 < \rho u(H, x_{\tau_1}) - c(\underline{\lambda} + \rho)$$

However, if L is indifferent at τ_{1-}, then $\dot{V}_{\tau_{1+}} = \rho u(L, x_{\tau_1}) - c(\underline{\lambda} + \rho) < u'_x(L, x_{\tau_{1+}})$ □

D Proofs for Section 3.4

Proof of Lemma 3. Suppose otherwise: at τ, the belief-path x_t jumps upwards forever. Since x_t is to jump up, by Lemma 4, it must be that the low type's effort jumps down at τ. Then, it must be that the low type is at least indifferent just before the jump, and after the jump, the low type is at most indifferent. If x_t is to remain constant after the jump, then it must be that both types choose $\underline{\lambda}$ after the jump happens. Then, we must have the following incentive compatibility constraint on the low type:

$$u(L, x_{\tau-}) - c \geq V_\tau^L = \frac{\underline{\lambda}}{\underline{\lambda} + \rho} u(L, x_\tau)$$

Then, for the high type, we have:

$$\begin{aligned}
&u(H, x_{\tau-}) - c = u(H, x_{\tau-}) - u(L, x_{\tau-}) + u(L, x_{\tau-}) - c \\
&\geq \frac{\underline{\lambda}}{\underline{\lambda} + \rho} u(L, x_\tau) + u(H, x_{\tau-}) - u(L, x_{\tau-}) \\
&= \frac{\underline{\lambda}}{\underline{\lambda} + \rho} u(H, x_\tau) + u(H, x_{\tau-}) - u(L, x_{\tau-}) - \frac{\underline{\lambda}}{\underline{\lambda} + \rho} \left(u(H, x_\tau) - u(L, x_\tau) \right) \\
&\geq \frac{\underline{\lambda}}{\underline{\lambda} + \rho} u(H, x_\tau) + u(H, x_\tau) - u(L, x_\tau) - \frac{\underline{\lambda}}{\underline{\lambda} + \rho} \left(u(H, x_\tau) - u(L, x_\tau) \right) \\
&> \frac{\underline{\lambda}}{\underline{\lambda} + \rho} u(H, x_\tau) = V_\tau^H
\end{aligned}$$

where the second inequality is due to decreasing differences and the assumption that the jump is upward. Hence, it follows that $\lambda^L_{\tau-} = \bar{\lambda}$.

Since both types choose the same effort at τ, the seller's posterior belief coincides with her prior belief. The prior belief changes continuously, and we must have that:

$$x_{\tau-} = \frac{\lambda^H_{\tau-}\mu_\tau}{\lambda^H_{\tau-}\mu_\tau + \lambda^L_{\tau-}(1-\mu_\tau)} = \frac{\bar{\lambda}\mu_\tau}{\bar{\lambda}\mu_\tau + \lambda^L_{\tau-}(1-\mu_\tau)} < \mu_\tau = x_\tau$$

which is not feasible for any feasible $\lambda^L_{\tau-}$. □

Lemma 8. *Suppose the low type chooses an interior effort before a downward terminal jump, then the high type chooses $\bar{\lambda}$ ($\underline{\lambda}$) before the jump.*

Proof. Suppose the low type is indifferent before the jump:

$$u(L, x_{\tau_2-}) - c = \frac{\bar{\lambda}}{\bar{\lambda}+\rho} u(L, x_{\tau_2}) - c\frac{\bar{\lambda}-\underline{\lambda}}{\bar{\lambda}+\rho} \Leftrightarrow$$
$$\rho u(L, x_{\tau_2-}) - c(\underline{\lambda}+\rho) = \bar{\lambda}\left(u(L, x_{\tau_2}) - u(L, x_{\tau_2-})\right)$$

Similarly, the high type given the above prefers $\bar{\lambda}$ at x_{τ_2-} whenever:

$$\frac{\rho u(H, x_{\tau_2-}) - c(\underline{\lambda}+\rho)}{\rho u(L, x_{\tau_2-}) - c(\underline{\lambda}+\rho)} > \frac{u(H, x_{\tau_2}) - u(H, x_{\tau_2-})}{u(L, x_{\tau_2}) - u(L, x_{\tau_2-})}$$

Consider the following function: $g(x) \equiv \left(u(H,x) - u(H, x_{\tau_2-})\right) / \left(u(L,x) - u(L, x_{\tau_2-})\right)$. Note that as $x \to x_{\tau_2-}$, $g(x) \to \frac{u'_x(H, x_{\tau_2-})}{u'_x(L, x_{\tau_2-})}$ and the desired inequality would hold by no-crossing property of $u(\cdot,\cdot)$. Since $x_{\tau_2} < x_{\tau_2-}$, to establish the result, it would suffice to show that $g(x)$ is increasing in x on $(0, x_{\tau_2-})$. Establishing this property is very similar to a no-crossing condition. Notice that $g(x)$ is increasing if

$$\frac{u(H,x) - u(H, x_{\tau_2-})}{u(L,x) - u(L, x_{\tau_2-})} \geq \frac{u'_x(H,x)}{u'_x(L,x)}, \forall x \in (0, x_{\tau_2-})$$

As argued before, the RHS is increasing in x, so that in the proximity of x_{τ_2-}, the inequality is satisfied:

$$\begin{aligned}\frac{u(H, x_{\tau_2-} + \Delta x) - u(H, x_{\tau_2-})}{u(L, x_{\tau_2-} + \Delta x) - u(L, x_{\tau_2-})} &= \frac{u'(H, x_{\tau_2-})}{u'(L, x_{\tau_2-})} + o(\Delta x) \\ &> \frac{u'(H, x_{\tau_2-})}{u'(L, x_{\tau_2-})} + \left(\frac{u'_x(H,x)}{u'_x(L,x)}\right)'_{x=x_{\tau_2-}} \Delta x + o(\Delta x)\end{aligned}$$

for $\Delta x < 0$. But then by the same proof as in no-crossing condition, it cannot be that the

LHS is smaller than RHS for any $x < x_{\tau_2 -}$.

□

Lemma 9. *Suppose the high type chooses $\bar{\lambda}$ at τ_2, and the low type is indifferent on an interval (τ, τ_2). Then, the high type chooses $\bar{\lambda}$ on the whole interval (τ, τ_2). In addition, if the low type chooses $\underline{\lambda}$ at τ_2, and x_t is continuous and strictly decreasing on (τ, τ_2), then the low type chooses $\underline{\lambda}$ on at interval (τ, τ_2) provided that $\bar{\lambda}$ is high enough.*

Proof. To satisfy the indifference condition for the low-type, it must be that on (τ, τ_2), the belief path satisfies:

$$\dot{x}_t = \frac{\rho u(L, x_t) - c(\underline{\lambda} + \rho)}{u'(L, x_t)}$$

Suppose by way of contradiction that the statement of the lemma is false. Then, some τ' exists, such that the high type's value crosses $u(H, x_{\tau'})$. Given (V-DE), we obtain that:

$$\dot{V}_{\tau'}^H = \rho u(H, x_\tau) - c(\underline{\lambda} + \rho) \text{ and}$$
$$\dot{u}(H, x_t) = u_x'(H, x_t)\dot{x}_t = u_x'(H, x_t)\frac{\rho u(L, x_t) - c(\underline{\lambda} + \rho)}{u'(L, x_t)} < \rho u(H, x_\tau) - c(\underline{\lambda} + \rho)$$

The inequality is due to the no-crossing condition. But then, the flow payoff increases slower than the value function. As we iterate backward in time, if an intersection point were to exist, the flow payoff would instantly get larger than the value again, which contradicts the existence of an intersection point in the first place.

Now, I prove the statement about the low type. Since x_t is assumed to be strictly decreasing on (τ, τ_2), it must be that either the high type is indifferent or chooses $\bar{\lambda}$. Whenever the high type is indifferent on an open interval, the proof is symmetric to the one for the high type. Suppose that the high type chooses the boundary effort. In this case, $\dot{x}_t = x_t(1 - x_t)(\underline{\lambda} - \bar{\lambda})$. To establish the result, it is sufficient to find $\bar{\lambda}$, such that:

$$x(1-x)(\underline{\lambda} - \bar{\lambda}) < \frac{\rho u(L, x) - c(\underline{\lambda} + \rho)}{u'(L, x)}, \forall x < x_{\tau_2}$$

Since in any equilibrium, the limit is reached with a downward jump, $x_{\tau_2} > 0$. In addition, by assumption $\rho u(L, \mu_0) - c(\underline{\lambda} + \rho) < 0$ so that the inequality is satisfied for all $x < \mu_0$. Hence, it is sufficient to have $\bar{\lambda}$, such that:

$$\bar{\lambda} > \underline{\lambda} - \min_{x \in [\mu_0, x_{\tau_2}]} \left\{ \frac{\rho u(L, x) - c(\underline{\lambda} + \rho)}{u'(L, x)} \frac{1}{x(1-x)} \right\}$$

$\square$

Proof of Proposition 5. Note that the market activity is slowing down with time whenever ${\Lambda^H}'(x) > 0$, since x_t is decreasing.

1) Let $\delta = u(H, x_{\tau-}) - u(L, \tilde{x})$. At $x_{\tau-}$, the sign of ${\Lambda^H}'(x_{\tau-})$ is determined by the sign of:

$$\frac{\tilde{x}}{x_{\tau-}} + \frac{x_{\tau-} - \tilde{x}}{\rho\left(u(H,\tilde{x}) - u(L,\tilde{x})\right)} u'(H, x_{\tau-}) \tag{3}$$

As $\delta \to 0$, $x_{\tau-} \to \tilde{x}$, $\lambda^H(x_{\tau-}) \to \underline{\lambda}$. Get a first order approximation of $x_{\tau-}$, from the its definition:

$$(x_{\tau-} - \tilde{x}) \approx -\frac{\rho}{\bar{\lambda} + \rho} \frac{\delta}{u'_x(H, \tilde{x})}$$

In limit, Expression (3) converges to $1 - \frac{1}{\bar{\lambda}+\rho} > 0$.

2) If $\bar{\lambda} \to \infty$, then $V^H(\bar{x}) \to u(H, \bar{x}) - c$, so that $x_{\tau-} \to \bar{x}$, which entails that the high type must choose approximately $\underline{\lambda}$ before the jump. We get again that ${\Lambda^H}'(x_{\tau-}) > 0$.

$\square$

E Construction of the Least Informative Equilibrium

As argued in the text, just before the jump the following must hold:

$$u(H, x_{\tau-}) - c \geq V^H_\tau = \frac{\bar{\lambda}}{\bar{\lambda} + \rho} u(H, \tilde{x}) - \frac{c(\bar{\lambda} - \underline{\lambda})}{\bar{\lambda} + \rho}$$

Note that because both types choose the same effort starting from τ, $x_\tau = \tilde{x}$ coincides with a prior belief μ_τ, so that $x_{\tau-} \in \left[\tilde{x}, \bar{\lambda}\tilde{x}/(\bar{\lambda}\tilde{x} + \underline{\lambda}(1-\tilde{x}))\right]$. If the inequality holds with a strict sign even at the right bound, that $\lambda^H_{\tau-} = \bar{\lambda}$, otherwise there is a unique $x_{\tau-}$ where the inequality binds and we can back out the effort by the high type right before the jump:

$$\lambda^H_{\tau-} = \underline{\lambda} \frac{x_{\tau-}(1-\tilde{x})}{\tilde{x}(1-x_{\tau-})}$$

Step 1. Suppose that $\lambda^H_{\tau-} = \bar{\lambda}$. In this case, the first candidate for the value function is given by the permanent choice of the upper bound $\bar{\lambda}$:

$$V^H(x) = \int_0^\infty e^{(-\bar{\lambda}+\rho)} \bar{\lambda} u\left(H, \frac{x}{x + (1-x)e^{(\bar{\lambda}-\underline{\lambda})t}}\right) dt$$

If $V^H(x) \leq u(H,x) - c, \forall x > x_{\tau-}$, then we have identified the correct solution and the high type chooses $\bar{\lambda}$ forever. Else, find that lowest x, where the suggested $V^H(x)$ crosses $u(H,x)-c$ [5]. Let x^1 be such an x.

Step 2. If $\lambda^H_{\tau-}$ is interior, then the first step is skipped, and we take x^1 to be $x_{\tau-}$. From x^1, we guess the effort by the high type to be a solution to:

$$\Lambda^{H\prime}(x) = \frac{\dot{\lambda}^H_t}{\dot{x}_t} = \Lambda^H(x)\left[\frac{1}{x(1-x)} + (\Lambda^H(x) - \underline{\lambda})\frac{u'_x(H,x)}{\rho u(H,x) - c(\underline{\lambda}+\rho)}\right],$$

with $\Lambda^H(x^1) = \bar{\lambda}$

By Picard–Lindelöf theorem, the above differential equation admits a unique solution in a neighborhood of $x^1 \in (0,1)$. We can concatenate the unique local solutions to a unique global solution on $[x^1, x^*)$ [6]

Let me now verify that in this solution (no matter what the value of x^1), $\Lambda^H(x) \underset{x \to x^*_-}{\rightarrow} \underline{\lambda}$, if $x^* < 1$. Suppose not, then:

$$\lim_{\delta \to 0} \Lambda^{H\prime}(x-\delta)\delta \rightarrow \frac{\Lambda^H(x) - \underline{\lambda}}{\rho} = -\frac{K}{\rho}, \text{for some } K > 0$$

implying that there exists some ξ, such that for all $\delta < \xi$, $\Lambda^{H\prime}(x-\delta) < -\frac{K}{2\rho\delta}$, which implies that:

$$\Lambda^H(x^*) < \Lambda^H(x^* - \xi) - \int_0^\xi \frac{K}{2\rho\delta} d\delta = -\infty$$

which is a contradiction. It then follows there exists some $x_0 = \Lambda^H(x^0)\mu_0/(\Lambda^H(x^0)\mu_0 + \underline{\lambda}(1-\mu_0)) < x^*$ whenever x^*. If $\Lambda^H(x) \leq \bar{\lambda}$, for all $x \in (x^1, x^0)$, then we are done. Else, there exists $\hat{x}$, where $\Lambda^H(x)$ crosses $\bar{\lambda}$ for the first time. In this case, move to Step 3.

Step 3. At $\hat{x}$ we fix $\Lambda^H(x)$ at the upper bound and start iterating the value function according to:

$$V^{H\prime}_x(x) = \frac{(\rho + \bar{\lambda})V^H(x) - \bar{\lambda}\rho u(H,x) + c(\bar{\lambda} - \underline{\lambda})}{x(1-x)(\underline{\lambda} - \bar{\lambda})}$$

with $V^H(\hat{x}) = u(H,\hat{x})$

[5] Such an x exists by the regularity assumption on $u(\cdot,\cdot)$ as will be argued further.

[6] Note that x^1 is bounded away from zero so that Picard–Lindelöf theorem can be applied. Indeed, either $x^1 \geq x_\tau > 0$, or x^1 is found from the first step. In this case, the intersection can only occur when the suggested $\Lambda^{H\prime}(x)$ is negative at $\Lambda^H(x) = \bar{\lambda}$. This can only occur if x^1 is sufficiently far away from 0. This gurantees that $\Lambda^{H\prime}(x)$ is Lipschitz continuous in Λ^H on a small enough subinterval of (x^1, x^*).

For the same reasons as in the previous step, there is a unique solution to this problem for all $(\hat{x}, 1)$. If the solution above is always below $u(H,x) - c$ for $x > \hat{x}$, then we are done. Otherwise, go back to Step 2 at a point where the intersection occurs for the first time.

Note that the regime change from $\bar{\lambda}$ to an interior λ can occur at most as many times as

$$\frac{1}{x(1-x)} + (\bar{\lambda} - \underline{\lambda})\frac{u'_x(H,x)}{\rho u(H,x) - c(\underline{\lambda}+\rho)}$$

changes sign. By the regularity assumption, $\rho u(H,x) - c(\underline{\lambda}+\rho) - x(1-x)(\bar{\lambda}-\underline{\lambda})u'_x(H,x)$ is analytic, which implies it changes its sign finitely many times on any closed interval. Hence, the construction converges.

Finally, note that given the construction of the value function, if $x^* < 1$, then $V^H(x) > \frac{\bar{\lambda}}{\bar{\lambda}+\rho}u(H,x) - \frac{c(\bar{\lambda}-\underline{\lambda})}{\bar{\lambda}+\rho}, \forall x < x_{\tau-}$. In particular, it implies that $V^H(x^*) - u(H,x^*) - c > \frac{1}{\bar{\lambda}+\rho}\left((\underline{\lambda}+\rho)c - \rho u(H,x^*)\right) = 0$. Hence, it must be that in the proximity of x^*, $\Lambda^H(x)$ is interior and abides the ODE of Step 2. This finalizes the construction of the least informative equilibrium.

F Proofs for Section 4.2

Proof. Proof of Lemma 6 The seller accepts the price given the belief x, whenever:

$$p \cdot \left[x(1-G^H(p)) + (1-x)(1-G^L(p))\right] - v_H x(1-G^H(p)) - v_L(1-x)(1-G^L(p)) \geq 0$$

Suppose that the above inequality holds at p, consider some $p' > p$ and suppose that the LHS crosses 0. At p', the derivative of LHS is:

$$\begin{aligned}
&\left[x(1-G^H(p')) + (1-x)(1-G^L(p'))\right] - x(p'-v_H)g^H(p') - (1-x)(p'-v_L)g^L(p') = \\
&\left[x(1-G^H(p')) + (1-x)(1-G^L(p'))\right] + \\
&+ (p'-v_L)(1-G^L(p'))\left(\frac{g^H(p')}{1-G^H(p')} - \frac{g^L(p')}{1-G^L(p')}\right) \\
&\geq (1-G^L(p'))\left[1 + (p'-v_L)\left(\frac{g^H(p')}{1-G^H(p')} - \frac{g^L(p')}{1-G^L(p')}\right)\right] \\
&> (1-G^L(p'))\left[1 + (v_H-v_L)\left(\frac{g^H(p')}{1-G^H(p')} - \frac{g^L(p')}{1-G^L(p')}\right)\right]
\end{aligned}$$

Hence, there can be no crossing at a higher p' provided that the adverse selection is constrained with

$$1 + (v_H-v_L)\left(\frac{g^H(p')}{1-G^H(p')} - \frac{g^L(p')}{1-G^L(p')}\right) \geq 0$$

.

Hence, the seller accepts all higher prices if p is accepted. In addition, note that the LHS is decreasing in x, given $p \in (v_L, v_H)$, which must hold at the intersections. So, the seller accepts fewer prices at a higher posterior belief x.

$\square$

Properties of $u(\cdot, \cdot)$ in the adverse selection set-up. Clearly, the first property is satisfied due to MLRP. Differentiating the expected utility with respect to x, we get:

$$u'_x(\omega, x) = -\mathbf{p}'(x) \int_{\mathbf{p}(x)}^{\bar{\varepsilon}} (\varepsilon - \mathbf{p}(x)) f(\mathbf{p}(x)) dG^{\omega}(\varepsilon) < 0$$

and is decreasing in ω given MLRP. Finally, note that by the proof of Lemma 5, to establish no-crossing condition, it is sufficient to have $u'_x(H, x)/u'_x(L, x)$ increasing.

$$\left(\frac{u'_x(H, x)}{u'_x(L, x)} \right)'_x = \left(\frac{\int_{\mathbf{p}(x)}^{\bar{\varepsilon}} (\varepsilon - \mathbf{p}(x)) dG^H(\varepsilon)}{\int_{\mathbf{p}(x)}^{\bar{\varepsilon}} (\varepsilon - \mathbf{p}(x)) f(\mathbf{p}(x)) dG^L(\varepsilon)} \right)'$$

The sign of the derivate above is determined by:

$${}_H[\varepsilon - \mathbf{p}(x) | \varepsilon > \mathbf{p}(x)] -_L [\varepsilon - \mathbf{p}(x) | \varepsilon > \mathbf{p}(x)] > 0$$

since MLRP is preserved after truncating both distributions at the same level.