# Sorting Stock through Sales: Inventory Turnover and Outlets

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November 1, 2024

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**Abstract**

A seller produces goods of two qualities: high and low. While the seller cannot observe individual product quality, consumers can, and they only purchase high-quality goods at higher prices. The seller becomes more pessimistic about the unsold inventory and reallocates it to a discount store. The key insight is that improving product sorting across stores requires attracting more consumers to a high-priced location so that fewer high-quality goods remain unsold and become available at a discount. This greater sorting enables the seller to charge a higher premium for high-quality goods, but it comes at the cost of reduced sales volume due to fewer consumers purchasing at higher prices. I extend this equilibrium relationship between sales volume, pricing, and product sorting to a model with a continuum of stores and provide additional insights into the seller's optimal strategy.

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# 1 Introduction

Many firms rely on outlet stores to mitigate the risks associated with excess inventory, especially when long design and production lead times coincide with short product lifecycles. [1] To free up valuable shelf space at flagship locations, underperforming products are often redistributed to outlets, where they are sold at discounted prices. [2] This approach not only frees up space but also protects flagship sales from "cannibalization." Since less popular products are more likely to remain unsold, consumers expect that desirable items will be harder to find at the outlet, incentivizing them to pay higher prices at the flagship store.

However, this reliance on outlet stores to offload unsold inventory creates several challenges for sellers. Because consumer demand determines which items remain unsold, shopping behavior directly influences current revenue and reshapes the product assortment across both stores. This reshaped assortment feeds back into future consumer preferences and shopping patterns, creating a tight link between prices, consumer behavior, and inventory composition at each store.

To address these challenges, this paper develops a theoretical model of store-level pricing and inventory management, focusing on how sellers can manipulate stock composition through outlet stores under conditions of extreme demand uncertainty. The model highlights the key trade-off between total sales volume and store differentiation. When the seller learns from unsold inventory and accounts for consumers' store choices, the equilibrium conditions induce an "upward-sloping demand curve" for the high-priced store, meaning that as prices increase, so does the customer share of a high-priced store—even when consumers are homogeneous.

To my knowledge, this is the first paper to study product sorting based on past purchasing behavior while explicitly accounting for consumer incentives in store selection.

## Two-Store Model: Key Mechanisms

To fix ideas, suppose a long-lived seller manages two stores: a high-priced flagship store and a low-priced outlet (Section 2). Each store holds a large number (a unit mass) of products, each of either high or low quality. The products are homogenous from the seller's perspective, but each consumer values a high-quality product more. Consumers engage in a directed

[1] For instance, Fisher and Raman (1996) provides a case study of Sport Obermeyer, a sportswear manufacturer that commits to production decisions about two years ahead, with 95% of its products being new designs. More recently, Consumer Technology Association (2023) emphasizes that a product lifecycle continues to shrink.

[2] This strategy is documented in Agrawal and Smith (2009), who also point out that while some items can be discounted within the same store, larger markdowns usually happen at other distribution channels.

search, choosing a shopping destination based on prices and the expected quality composition. All else equal, they favor stores with a higher proportion of high-quality goods since they randomly inspect one product at their chosen location. Upon inspection, a consumer learns the product's quality and decides whether to purchase it at the price posted for the store. The prices are set so that the consumers only purchase high-quality goods at the flagship but buy both high- and low-quality goods at the outlet. Additionally, assume that the seller's production costs are sufficiently high, so it is too expensive for the seller to throw any of her goods away.

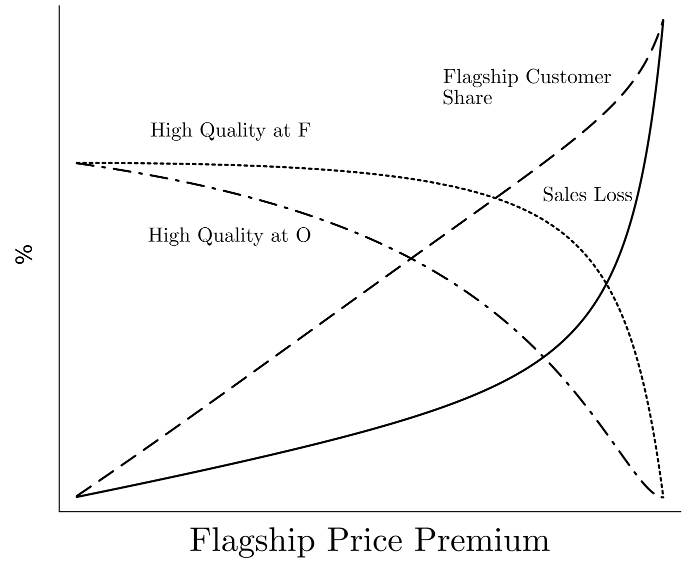
Without the ability to directly distinguish between high- and low-quality products, the seller relies on consumers' purchasing decisions to effectively 'sort' the products for her. I focus on the stores' steady-state (long-run) quality composition, which emerges as follows. At the start of each period, a constant flow of short-lived consumers arrives at the market. Consumers then choose where to shop and whether to purchase the product they inspect. At the end of the period, the seller restocks both stores back to their full capacity (which remains fixed over time).

Restocking the outlet with unsold flagship inventory signals lower product quality at the outlet to consumers. This is the key sorting mechanism: as consumers skim off high-quality goods at the flagship, the outlet is consistently restocked with products of lower consumer value. This allows the seller to offer a markdown and offload her low-value items at the outlet without diverting every customer from the flagship location, as they are willing to pay a premium for its superior quality composition.

The model's central insight reveals a limitation of the seller's sorting mechanism: as the flagship's relative quality premium increases, more consumers shop at the flagship, the product quality at both stores declines, and the seller loses potential sales volume (summarized on Figure 1). Intuitively, as the seller acts on the information from goods remaining unsold to learn more, the seller needs to sell less. Specifically, to enhance the flagship's quality premium, the seller accelerates the purchases of high-quality items at the flagship, reducing the number of such products shipped to the outlet. This requires attracting a larger share of consumers to the flagship. Consequently, the equilibrium restrictions induce an upward-sloping "demand curve" for the flagship store, even though consumers are homogenous.

While greater quality differentiation allows the seller to charge a higher flagship price, it comes at a cost: as more consumers shop at higher prices, total sales volume declines. One reason is the direct effect: as consumers purchase only a fraction of goods (high-quality items) they inspect at the flagship, more consumers will fail to make a purchase. A second effect is due to the subsequent change in the quality composition. As lower-quality products are purchased less intensively, the overall inventory turnover rate falls. The stock is refreshed more slowly, and at the steady state, the overall quality composition gets worse across both stores.

*Figure 1: Equilibrium Limitations on Seller's Pricing Strategy*




*Note*: the figure summarizes the equilibrium relationship of the two stores model key variables (see Section 2). The sales loss is computed relative to the maximal potential per-period sales volume.

## Going Beyond Two Stores

Given the insights from the two-store model, an intriguing question arises: what could the seller achieve if not constrained by only two stores and instead had the flexibility of utilizing multiple locations? To explore this possibility, I study inventory replenishment in continuous time.

I then pursue the question of product screening through multiple locations to its limit by considering a model of infinitely many (a continuum of) different stores (Section 3). In this model, the seller selects the joint distribution of prices, consumers, and steady-state quality compositions across the locations, subject to the same equilibrium constraints. In particular, the seller can flexibly determine how many stores serve as outlet locations, offering products at a low price.

First, I characterize the potential steady-state equilibria the seller could sustain with some prices. I show that equilibria outcomes take a simple form: there exists a threshold that divides into two groups. On one side of the threshold are all high-priced locations, where consumers purchase only high-quality goods; on the other side, consumers shop only at outlet locations, offering a low price that compels them to purchase products of any quality. Additionally, the payoffs of the players are determined by the quality composition of this threshold, which reduces the dimensionality of the seller's problem to a choice of a single parameter.

Similar to the two-store model, the seller faces a trade-off: she can increase store differentiation (worsen the quality composition at the outlet threshold) only by reducing her total sales volume. When the seller must rely on outlet locations to dispose of low-quality goods, Theorem 2 also shows the seller maintains a nontrivial measure of such locations. Without these outlet locations, low-quality products would flood the entire stock, ultimately driving sales to zero.

Building on the trade-off between outlet quality and sales distribution, I further show that as the value of high-quality products rises, the seller lowers the outlets' quality composition to raise prices at other store locations (Proposition 5). Similarly, as the seller becomes more certain about the appeal of her products, she screens product types more aggressively. With increased confidence in product quality, the value of consumers' expertise in product assessment diminishes, reducing their leverage to command high information rents.

In addition, I show in Theorem 3 that when the inventory shipments occur infinitely often, the seller benefits from having infinitely many store locations. A continuum of stores allows for more nuanced quality-price differentiation, minimizing the risk of misclassifying products for reduced pricing and thus enabling the seller to capture a greater share of consumer surplus.

## Additional Results and Model Limitations

While the baseline model offers initial insights into product assortment, it leaves out many important aspects of retail business. I address some of these limitations within this paper by appropriately extending the model. Others remain outside the scope of the paper and offer interesting venues for future research. I discuss the most prominent ones below.

As a first extension of the model, I allow for direct disposal of inventory. In contrast to the original model, where the seller frees up shelf space solely through sales, retailers sometimes choose to destroy unsold products instead of offering discounts. [3] In Section 4.1, the seller can choose the rate at which the unsold products are destroyed. Direct disposal is costly, interpreted as the seller bearing either a disposal fee or a replacement (production) cost. I show the seller uses only one disposal method at the optimum (Proposition 6). When disposal costs are high, the seller relies on outlets to dispose of the unsold inventory; when

[3] See, for instance, the recent investigation (Satariano, 2021) into Amazon's practice of unsold inventory destruction.

low, the seller discards inventory directly to maintain high prices across all store locations.

Another extension introduces multiple quality tiers (Section 4.3), acknowledging the complexity of real-world product differentiation. Here, the product can take multiple values for consumers rather than simply high or low. I verify in Proposition 7 that the main results can be adapted to this richer framework.

The final extension examines consumer preference heterogeneity (Section 4.4). As existing literature often emphasizes the role of outlet stores in consumer segmentation,[4] I introduce vertical differentiation in consumers' willingness to pay for high-quality products. This extension reveals how product- and consumer-type screening interact. I show that consumers self-sort along the replenishment chain with earlier locations sustaining higher quality and prices and attracting higher consumer types (Proposition 8). Outlet locations thus serve a dual role of managing product assortment and screening consumer types in this version of the model.

This paper examines how sellers leverage outlet stores for product assortment management while recognizing their broader functions, such as market segmentation. To focus on the consumer's role in product sorting, the model abstracts away from some crucial aspects of retail operations. One notable omission is time-based quality depreciation, such as seasonality, which significantly complicates inventory management. Time depreciation enhances opportunities for quality differentiation but also heightens the importance of rapid inventory turnover. Additionally, the model limits the seller's use of sales performance data to a fixed strategy of reallocating products in a single direction down the line of stores. While this assumption may reflect practical constraints—where more granular or dynamic data usage may be prohibitively costly[5]—it leaves many questions about optimal inventory management open. For example, future research could explore whether more flexible approaches to inventory reallocation could provide further benefits for the seller.

## Related Literature

The paper contributes to a broad literature on directed search with adverse selection. In particular, many papers (see Guerrieri, Julien, and Wright (2017) for a review) study the sorting and segmentation of different types of agents in the general equilibrium context. In these models, sellers have superior information about the quality of goods and choose which contracts to seek in the market. Buyers, in turn, decide which contracts to offer. In equilibrium, a distribution of contracts emerges that facilitates screening, where buyers balance

[4] See, for example, Coughlan and Soberman (2005), Ngwe (2017), Li (2023).

[5] For example, Caro, Babio, and Peña (2019) highlights that Zara offers around 8,000 products annually, suggesting more nuanced design-level inventory/markdown decisions may be impractical.

the terms of contracts against the probability of matching, thereby mitigating information asymmetry.

My model shares key themes with this literature: the probability that a product is of high quality at a given location can be interpreted as a matching function. Higher-priced locations provide a greater chance of matching with high-quality goods, allowing the seller to incentivize consumers to shop at different prices. Guerrieri, Julien, and Wright (2017) also emphasizes that the relationship between demand and the price in the equilibrium is ambiguous, depending on the matching function.[6] Unlike traditional models focused on buyer-seller matching, my model shifts the focus to product-specific matching and examines an optimal market outcome design by a seller. In my model, the seller actively manages future sales probabilities by adjusting inventory across locations. This strategy allows the seller to exploit adverse selection in a controlled way, optimizing the product mix across store types and influencing consumer behavior.

Information aggregation on product quality plays a key role in my model. While other models (e.g., Lauermann and Wolinsky (2016)) emphasize how well prices aggregate information, my approach introduces inventory reallocation as a crucial mechanism for shaping the quality composition of products across stores. Prices alone do not fully convey all the necessary information; instead, the seller filters product assortments through locations and price points, allowing the quality of goods to be sorted along by consumer behavior.

As the seller in my model learns about the product through sales, the paper is also related to the literature on two-sided learning. Bergemann and Välimäki (1997), Bergemann and Välimäki (2000), Bergemann and Välimäki (2006), Bonatti (2011) explore market environments where a firm is initially uninformed about the quality of its product. Over time, the market gradually learns the product type, with the speed of learning increasing in the speed of sales, contrary to the trade-off I identify in my paper.

The structure of inventory shipments in my model parallels the hierarchical labor organization framework presented by Garicano (2000). In their model, firms learn about task difficulty by passing tasks between workers with different skill levels. Similarly, in my model, the seller learns about product quality by filtering goods through various price points. However, while Garicano's model focuses on task learning within firms, my model operates in a different context: learning here is driven by consumer behavior, with buyers revealing information through their purchasing decisions. Crucially, this learning process requires incentivizing consumers through strategic pricing.

The paper also contributes to operations research and marketing literature, studying in-

[6] Although the most widely used function does not generate the positive relationship between the price and the demand.

ventory management under uncertain demand. For some of these papers, the seller learns about the demand for her product with time and updates inventory orders from the production accordingly (see Silver, Pyke, and Thomas (2016) for a review). Other papers assume that current sales affect future demand (either directly—through "contagious' demand, or indirectly— through seller's learning) or when current sales are information implying an exploration/exploitation problem for the seller (e.g., Hartung (1973), Petruzzi and Monahan (2003), Caro and Gallien (2007)). Importantly, these papers abstract away from how the seller's decisions affect consumers' incentives to shop at any specific location, which is a crucial part of my paper. Ngwe (2017) adopts a similar model of inventory replenishment for an empirical analysis of market segmentation using outlet locations but does not account for the two-sided learning that I explore in my paper. In particular, in Ngwe (2017), the seller can accurately price her products, and consumers can easily find their preferred product at the location of their choice.

The paper also contributes to the broad literature on revenue management (see Gallego and Van Ryzin (1994), Den Boer (2015), Elmaghraby and Keskinocak (2003), Board and Skrzypacz (2016), Dilme and Li (2019)). These papers study the optimal dynamic prices to segment the different consumers when they can strategically delay purchases. In my paper, the prices for a product change dynamically but not due to product (rather than consumer) heterogeneity.

Methodologically, this paper contributes to the literature on steady-state mechanism design, as in Madsen and Shmaya (2024) and Baccara, Lee, and Yariv (2020).

# 2 Two-Store Model

The section outlines a two-store model where a seller manages a high-priced flagship store and a low-priced outlet to sell products of uncertain quality: high or low. Unable to distinguish the product quality, the seller relies on past purchases to sort products across the stores. Specifically, she replenishes the outlet with unsold inventory from the flagship, which is more likely to be of lower quality. The analysis focuses on the steady-state equilibrium, where consumers optimally choose where to shop while shaping the long-run product assortment across the stores. In Theorem 1, I formalize the paper's key insight on how buyers' choices limit the seller's strategy: raising the flagship price requires drawing more consumers to the flagship, where they make fewer purchases. Hence, the model underscores the trade-off between sales volume and product differentiation. The seller benefits from operating both stores when consumers place a high value on quality or when high-quality goods are produced frequently enough. The findings of this section set the stage for my analysis of the optimal pricing strategy in a more complex setting of Section 3.

## 2.1 Model

Suppose a single long-lived seller (female) manages two store locations: a flagship store and an outlet. Each store holds a continuum of products of mass 1. The quality of any particular product can be either high or low. When produced, each product is of high quality with probability $\pi$. The seller perceives all goods as homogeneous, unable to distinguish their quality. Consumers (males) derive utility $v^h$ from high-quality products and $v^l$ from low-quality products, with $v^h > v^l > 0$. Unlike the seller, consumers can identify the product's quality upon finding it in the store. Time is discrete and runs over an infinite horizon, $t \in \{1, 2, \dots\}$. Each period, a mass $\lambda \in (0, 1)$ of short-lived consumers arrives at the market.

As the seller does not observe the quality of the products, she cannot directly set prices based on product quality. Instead, she can charge different prices at the two stores. Specifically, prices at both locations remain constant over time, with the seller charging $p^f$ at the flagship store and $p^o$ at the outlet.

**Inventory replenishment.** I assume the seller follows a predetermined inventory policy, which, combined with consumer purchasing decisions, shapes the quality composition at both stores. After consumers complete their purchases, the seller replenishes both stores to full capacity. Replenishment occurs sequentially: first, the outlet is restocked with unsold items from the flagship's inventory, chosen randomly. Once the outlet is restocked, the seller replenishes the flagship store with new products from the production plant. I refer to this inventory policy as *sequential replenishment*.[7] For the two-store model, I assume the seller faces prohibitively high costs for directly disposing of her goods.[8]

**Consumer Behavior**. The sales of each product type are determined by consumer behavior. I assume that upon arrival, consumers do not observe the current calendar time and choose which location to shop at based on each store's price and their expectations about the quality composition. Their shopping strategy remains constant over time, with $\sigma$ denoting the share of consumers who choose the flagship store.

Each buyer randomly draws a product from the current stock at their selected location and learns its quality. I assume these product draws occur simultaneously, with each consumer drawing a unique product. Therefore, if a store $i$ holds a share $q^i$ of high-quality goods, each shopper of this store draws a high-quality product with probability $q^i$.[9] Once the product's

[7] In Section 2.2.2, I compare this policy with a direct replenishment strategy, where new inventory is ordered from the production plant for both locations.

[8] As I explain in the continuous version of the model in Section 4.1, this assumption can be interpreted as restricting the model's parameters so that the seller's production costs are sufficiently high.

[9] This assumption is reasonable if consumers are small enough relative to the stock.

quality is known, the consumer decides whether to purchase it based on the store's price. If he purchases a product of quality $\omega \in \{l, h\}$ at price $p$, his payoff is $v^\omega - p$. For simplicity, I assume that buyers always make a purchase when indifferent.

**Restriction on Prices**. Given the motivation for the sequential replenishment and to ease the exposition, I restrict attention to the case where the flagship's price is high $p^f \in (v^l, v^h)$, so that the consumers only purchase high-quality there. At the outlet, the price is set just to encourage the purchases of both product types: $p^o = v^l$. In Appendix B, I discuss other possible prices and verify it is without loss for the seller's optimality to restrict prices this way.

**Quality Composition Evolution.** The quality composition at each location evolves according to the sequential replenishment policy, given that consumers' purchasing decisions are rational. Suppose at the beginning of period $t$, the proportion of high-quality goods at each store $i \in \{f, o\}$ is given by $q_t^i$.

Consider the outlet first. At the outlet, consumers purchase any product type they find (following the tie-breaking rule). Therefore, total sales at the outlet in any given period equal its consumer flow $(1 - \sigma)\lambda$, with a share $q_t^o$ of these sales being of high quality. To replenish the outlet, the seller ships inventory from the flagship equal to the total outlet sales, $(1-\sigma)\lambda$. The share of high-quality goods in the shipments is the proportion of high-quality goods in the flagship's after-sales remaining inventory, denoted $q_{t,a}^f$. Thus, the total change in the mass of high-quality items[10] at the outlet is

$$\Delta q_t^o = q_{t,a}^f(1 - \sigma) - q_t^o(1 - \sigma)$$

Next, consider the evolution of the quality composition at the flagship store. Since the flagship's price is high, consumers only purchase high-quality goods. The total mass of purchases at the flagship equals $\sigma q_t^f$, the mass of consumers who find a high-quality product. The flagship gets restocked to full capacity once consumer purchases and shipments to the outlet are complete. Hence, the total mass of new inventory ordered from the production plant to the flagship equals the mass of total sales at both stores in period $t$, which is $q_t^f\sigma + (1 - \sigma)$. A fraction $\pi$ of these new items is of high quality. Hence, the change in the flagship's share of high-quality items is given by

$$\Delta q_t^f = \pi(q_t^f\sigma + (1 - \sigma)) - \sigma q_t^f - q_{t,a}^f(1 - \sigma)$$

**Steady-state equilibria in the consumers' game**. For my analysis, I use the steady-

[10]Given the stock of either store is normalized to one, the change in the proportion of high-quality goods at any store coincides with the change of their mass.

state quality composition as part of the solution concept. The quality composition $(q^f, q^o)$ is a steady state *induced by consumers' shopping strategy* $\sigma$ when the proportion of high-quality goods at both stores remains constant over time: $\Delta q_t^i = 0$ for $q_t^i = q^i$.

For simplicity, I assume that once consumers decide where to shop, the market immediately reaches a steady state. On the seller's side, one can interpret this assumption as the seller being infinitely patient but highly averse to profit fluctuations over time in the long run. On the consumer's side, this assumption can be interpreted as the consumers' beliefs being dominated by the steady-state quality composition.

Consumers select their shopping strategy $\sigma$ to maximize their expected payoff given the prices and the steady-state quality composition. At the flagship, consumers expect to find a high-quality item with probability $q^f$ and receive a payoff of $v^h - p^f$. If they find a low-quality item, they leave the market with no purchase. At the outlet, they find high-quality goods with probability $q^o$ and receive $v^h - v^l$. With probability $1 - q^o$, they find a low-quality item, purchase it, and get a zero payoff. Therefore, a consumer's expected payoff at the flagship with probability $\sigma$ is

$$V^B(p^f, \sigma, q^f, q^o) = \sigma q^f (v^h - p^f) + (1 - \sigma) q^o (v^h - v^l)$$

The shopping strategy $\sigma$ and steady-state quality composition $(q^f, q^o)$ form *a steady-state equilibrium in the consumers' game* if $(q^f, q^o)$ is induced by $\sigma$ and $\sigma$ is consumer-optimal, given the prices and expected quality at each store at the steady state. Let $\mathcal{E}p^f$ denote all possible equilibria in the consumers' game given the flagship price $p^f$.

**Seller's Problem**. The seller chooses the flagship price $p^f$ and any steady-state equilibrium $(\sigma, q^f, q^a) \in \mathcal{E}p^f$ to maximize her per-customer steady-state profit flow in both stores. At the flagship, customers only purchase at a price $p^f \in (v^l, v^h)$ when they find a high-quality item, which occurs with probability $q^f$. Thus, when a share $\sigma$ of consumers select the flagship, the seller's steady-state flagship revenue is $p^f q^f \sigma$. Consumers purchase both product types at the outlet, where the price is $p^o = v^l$, yielding outlet revenue of $v^l(1 - \sigma)$. Therefore, the seller's total steady-state profit across both stores is

$$V^S(p^f, \sigma, q^f) = p^f q^f \sigma + v^l (1 - \sigma)$$

## 2.2 Sorting Through Sales: Equilibrium Analysis

This section analyzes the two-store model and presents the main insight of the paper in Theorem 1. I show that sequential replenishment allows the seller to sustain an interior equilibrium, where she can attract consumers to both the flagship store and the outlet. In this equilibrium, as the flagship price rises, more consumers shop at the high-priced store, but the quality composition at both stores and per-period sales volume decrease. This reveals the seller's trade-off between sales rates and store differentiation. Moreover, Theorem 1 shows that due to the interaction between prices, product sorting, and consumer shopping behavior, the seller faces an upward-sloping demand curve when setting the flagship price.

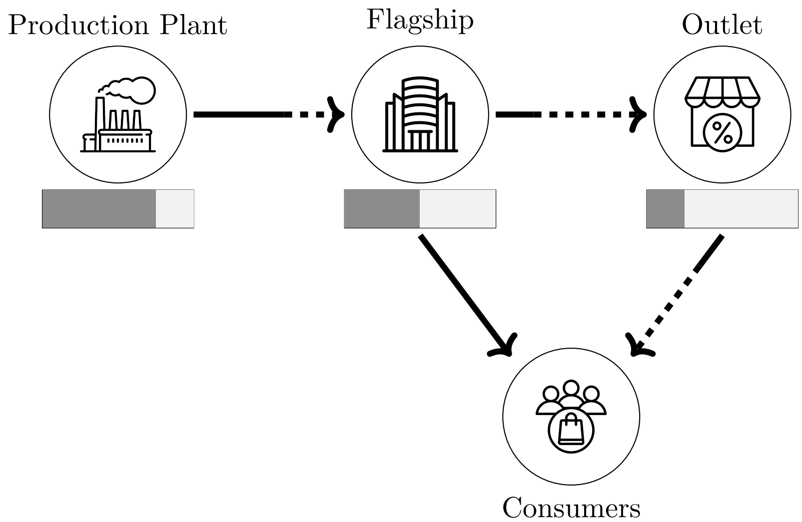
### 2.2.1 Induced Steady State

In this section, I fix a consumer strategy $\sigma$ and analyze the steady-state quality composition it induces. I then use the insights gained here to characterize the equilibrium constraints on the flagship price and consumers' shopping strategy in Section 2.2.2.

First, I demonstrate that if the seller exclusively relies on the flagship store ($\sigma = 1$), the absence of direct disposal causes low-quality items to overaccumulate. Over time, they displace the high-quality ones, leading to a point where consumers can no longer find the goods worth purchasing at premium prices, which drives long-run sales to zero.

In contrast, when some consumers opt to shop at the outlet ($\sigma < 1$), each consumer strategy $\sigma$ induces a unique steady-state quality composition across stores, which is positive. In this steady state, the flagship retains a higher proportion of high-quality products than the outlet. Sequential replenishment thus enables the seller to partially sort product quality across the two stores, even though she is initially uninformed. Figure 2 illustrates these key features of the sequential replenishment for the case when $\sigma < 1$.

*Figure 2: Sequential Replenishment: Induced Steady State*




*Note*: The figure illustrates the sequential replenishment rule, with arrows indicating the direction of inventory flow. The arrows are divided according to the proportion of high- and low-quality items in the inventory flows: solid lines represent high-quality items, and dashed lines represent low-quality items. Rectangles represent the stock of a respective location. Dark shading represents high-quality products, while light shading represents low-quality ones.

Moreover, both steady-state average quality levels decrease as $\sigma$ increases. Intuitively, when a larger share of consumers shop at the flagship, fewer low-quality goods are purchased, causing these goods to remain in stock longer and occupy more shelf space in both stores. Importantly, this adverse selection impacts the outlet more severely: as more high-quality items are purchased at the flagship, fewer of them are shipped to the outlet. Lemma 1 formalizes these key properties of the induced steady-state quality composition.

**Lemma 1.** *If $\sigma = 1$, then the induced state is any $(q^f, q^o) \in [0,1]^2$, such that $q^f = 0$. For every $\sigma < 1$, there exists a unique steady-state quality composition, $(\mathbf{q}^f(\sigma), \mathbf{q}^o(\sigma))$. Moreover,*

*(i)* *both $\mathbf{q}^f(\cdot)$ and $\mathbf{q}^o(\cdot)$ are decreasing,*

*(ii)* $\mathbf{q}^f(\cdot)/\mathbf{q}^o(\cdot)$ *is increasing,*

*(iii)* $\mathbf{q}^f(0) = \mathbf{q}^o(0) = \pi$,

*(iv)* $\mathbf{q}^f(\sigma) > \mathbf{q}^o(\sigma) > 0$ *if* $\sigma \in (0,1)$.

I now go over the key steps of the proof for Lemma 1 and first consider the steady-state quality composition for the outlet. If at least some consumers shop there ($\sigma < 1$), the outlet holds the same quality composition as the flagship's unsold inventory. Indeed, given the quality evolution implied by sequential replenishment, the outlet's quality composition remains constant, $\Delta q_t^o = 0$, whenever $q_t^o(1-\sigma)\lambda = q_{t,a}^f(1-\sigma)\lambda$. Hence, either the outlet's quality composition matches the share of high-quality goods in the flagship's unsold inventory $q^o = q_a^f$, or no consumers shop at the outlet $\sigma = 1$. For the special case, any outlet's quality composition is consistent with a steady state (as there is no inventory movement in the outlet, any initial quality composition remains constant)

It is now easy to see that the sequential shipments allow the seller to support a quality premium for the flagship relative to the outlet. As consumers only pick out high-quality goods at a high flagship price $p^f \in (v^l, v^h)$, at the end of the period, the unsold inventory of the flagship holds more of the low-quality items. I now show this formally. Given the high prices, the total sales at the flagship equal the total mass of consumers who inspected a high-quality product, $q^f\sigma\lambda$. The total remaining stock at the flagship is $1 - q^f\sigma\lambda$, while the remaining stock of high-quality goods is $q^f(1-\sigma\lambda)$. This results in the after-sales proportion of $q_a^f = q^f(1-\sigma\lambda)/(1-q^f\sigma\lambda)$. This proportion is strictly below the flagship's steady-state quality composition $q^f$ whenever $1 > q^f > 0$ and a positive share of consumers shop at the flagship ($\sigma > 0$).

I now characterize the flagship's steady-state quality composition. To do so, I equate the total outflow of high-quality items to their inflow ($\Delta q_t^f = 0$):

$$\underbrace{\sigma\lambda q^f}_{\substack{\text{high quality good sales}\\ \text{in the flagship}}} + \overbrace{\frac{q^f(1-\lambda\sigma)}{1-q^f\lambda\sigma}}^{\text{after-sales proportion of high quality} q_a^f} \underbrace{(1-\sigma)\lambda}_{\substack{\text{inventory shipment}\\ \text{to outlet}}} = \pi \underbrace{\left(q^f\sigma\lambda + (1-\sigma)\lambda\right)}_{\substack{\text{total inventory order}\\ \text{from production}}} \quad (1)$$

In Lemma 7 in Appendix A, I verify that any $\sigma \in [0,1]$, the above equation uniquely defines the flagship's steady-state quality composition $\mathbf{q}^f(\sigma)$.

In addition, the more consumers shop at the flagship, the worse its induced steady-state average quality (see Lemma 7 in Appendix A). To illustrate, consider the two corner cases. When $\sigma = 1$, Equation (1) is satisfied whenever $q^f = \pi q^f$, which only holds if $q^f = 0$. As low-quality items do not get a chance of being purchased, eventually, they overtake the flagship's stock. On the other hand, if all consumers shop at the outlet ($\sigma = 0$), then Equation (1) becomes: $q^f = \pi$.

Consequently, the outlet's steady-state quality composition is also unique for any shopping strategy $\sigma \in [0,1)$ and is given by the after-sales quality composition of the flagship at the steady state: $\mathbf{q}^o(\sigma) = \mathbf{q}^f(\sigma)(1-\sigma\lambda)/(1-\sigma\lambda\mathbf{q}^f(\sigma))$.

We can now show the average quality of the outlet is affected more by the increased adverse selection: $\mathbf{q^f}(\sigma)/\mathbf{q^o}(\sigma)$ is increasing in $\sigma$. To verify this, recall that, at the induced steady state,

$$\frac{\mathbf{q}^f(\sigma)}{\mathbf{q}^o(\sigma)} = \frac{1 - \mathbf{q}^f(\sigma)\lambda\sigma}{1 - \lambda\sigma}$$

Differentiating the above with respect to $\sigma$, we get:

$$\frac{\partial \mathbf{q}^f(\sigma)/\mathbf{q}^o(\sigma)}{\partial \sigma} = \frac{-\lambda\sigma(1-\lambda\sigma)\frac{\partial \mathbf{q}^f(\sigma)}{\partial \sigma} + \lambda(1 - \mathbf{q}^f(\sigma))}{(1-\lambda\sigma)^2} > 0$$

where the inequality follows from the fact that the flagship's proportion of high-quality goods decreases with the share of flagship shoppers: $\frac{\partial \mathbf{q}^f(\sigma)}{\partial \sigma} < 0$ (due to Lemma 7 in Appendix A).

### 2.2.2 Steady State Equilibria

Thus far, I have considered how a given shopping strategy $\sigma$ affects the quality composition. Now, to construct equilibria in the consumers' game, I require the shopping strategy to be optimal given the prices and the steady-state quality composition at both stores.

Given Lemma 1, the following two corner equilibria can arise for any flagship price $p^f$. First, the outlet may cannibalize the flagship fully. In this case, all consumers choose the outlet ($\sigma = 0$), and both stores hold the highest possible share of high-quality products, determined by the production plant's average: $q^f = q^o = \pi$. Since both stores offer the same expected quality but the outlet has a lower price, all consumers prefer to shop at the outlet, making $\sigma = 0$ optimal.

In the second corner equilibrium, the seller makes zero sales. Indeed, suppose all consumers choose the flagship $\sigma = 1$. By Lemma 1, the flagship's proportion of high-quality goods in any induced steady state is 0. For the outlet, any initial quality composition is at an induced steady state, as it receives no visitors and experiences no inventory movement. In particular, to sustain this corner equilibrium, we can choose $q^o = 0$ and make consumers indifferent between the two stores.

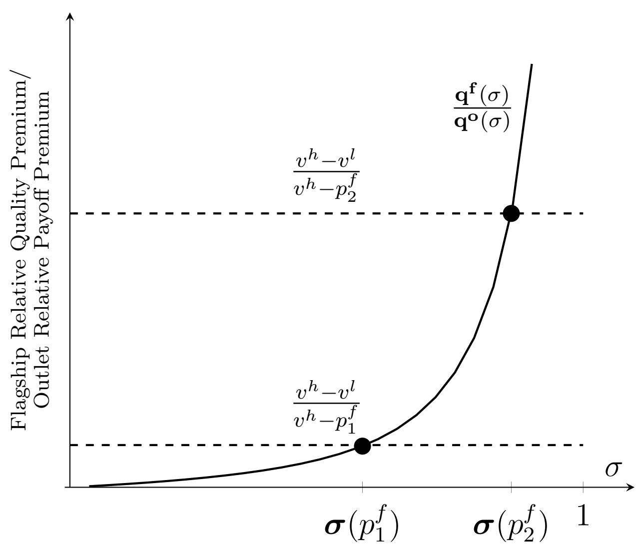
However, every flagship price $p^f \in (v^l, v^h)$ also allows for an interior equilibrium in which both stores serve some customers. I now construct such an interior equilibrium. At this equilibrium, each buyer must be indifferent between the two stores. Otherwise, all consumers would prefer to shop at the same store. Hence, for any given flagship price $p^f$, the buyer's shopping strategy $\sigma$ must be such that the difference in the induced steady-state quality compositions $(\mathbf{q^f}(\sigma), \mathbf{q^o}(\sigma))$ exactly offsets the price differential between the stores:

$\mathbf{q^f}(\sigma)(v^h - p^f) = \mathbf{q^o}(\sigma)(v^h - v^l)$ or

$$\frac{\mathbf{q^f}(\sigma)}{\mathbf{q^o}(\sigma)} = \frac{v^h - v^l}{v^h - p^f}$$

Since $\mathbf{q^f}(\sigma)/\mathbf{q^o}(\sigma)$ is increasing in $\sigma$, one can find a unique share of flagship shoppers $\boldsymbol{\sigma}(p^f)$ which sustains this equilibrium for a given flagship price $p^f$. Interestingly, in this equilibrium, the share of consumers shopping at the flagship increases with the flagship price $p^f$ (Figure 3). The outlet's payoff premium (conditional on purchase) rises as the flagship price increases from $p_1^f$ to $p_2^f$.

*Figure 3: Steady State Equilibrium Shopping Strategy*




*Note*: the figure plots the quality composition at the steady state for different shopping strategies $\sigma$ of the buyers and illustrates the equilibrium shopping strategy for different choices of the flagship price $p^f$.

Therefore, to preserve the flagship store's attractiveness, the average quality at the outlet store must fall relative to that of the flagship. A higher degree of quality differentiation between the two stores can only be achieved if high-quality goods are purchased faster at the flagship store, requiring more consumers to shop there.

Consequently, the total sales volume $\lambda[\sigma q^f + (1-\sigma)]$ at this interior equilibrium decreases in the flagship price. As the flagship price rises, more consumers are pushed towards the flagship store in the equilibrium, leading to a decline in total sales due to two effects. The first is a direct effect: consumers do not purchase all product types they find at the flagship (in contrast to the outlet). The second effect is due to the subsequent deterioration of the flagship's steady-state quality composition (due to Lemma 1). As more consumers shop at the flagship, its average quality declines and it becomes more difficult for flagship shoppers to find high-quality products worth buying at a high price, further driving sales down. I summarize these key takeaways in Theorem 1.

**Theorem 1.** *With sequential replenishment, for every flagship price $p^f \in (v^l, v^h)$, there exists a unique interior steady-state equilibrium exists in the consumers' game, in which consumers shop at both stores with positive probability ($\sigma \in (0,1)$). Moreover, in this equilibrium, if price $p^f$ increases*

*i) the customer share of the flagship store $\sigma$ rises,*

*ii) the quality composition at both stores $(q^f, q^o)$ gets worse,*

*iii) the relative quality-differentiation between stores $q^f/q^o$ rises,*

*iv) and the total steady-state per-period sales $\lambda[\sigma q^f + (1-\sigma)]$ decrease.*

Note that Theorem 1 can be interpreted as a version of the Veblen effect. As the flagship's price rises, more consumers are attracted to the flagship's store, as the high-quality becomes rarer.

**Direct replenishment.** Before proceeding to analyze the seller's problem in the next section, I will first discuss the benefits of sequential replenishment by comparing it to an alternative inventory policy.

Suppose that after the consumer makes their purchases, the seller orders new inventory to replenish both stores directly from the production plant. I call this *direct replenishment*. Unlike sequential replenishment, direct replenishment from the production plant for both stores would allow for no interior equilibrium, as it fails to leverage information from unsold inventory.

Suppose that after the consumer makes their purchases, the seller orders new inventory to replenish both stores directly from the production plant. In this case, when the outlet makes sales in the amount of $\lambda(1-\sigma)$, its inflow of high-quality products becomes $\pi\lambda(1-\sigma)$—as all its incoming inventory comes from the production plant. Consequently, the outlet's quality composition evolves as follows: $\Delta q_t^o = \pi\lambda(1-\sigma) - q_t^o\lambda(1-\sigma)$. The flagship only gets restocked for its own sales, so the new inventory is ordered in the amount of $q_t^f\lambda\sigma$, with a share $\pi$ of these being of high quality. Therefore, the flagship's quality composition evolves according to $\Delta q_t^f = \pi\lambda\sigma q_t^f - \sigma\lambda q_t^f$.

**Proposition 1.** *With direct replenishment, no interior equilibrium in the consumers' game exists. In addition, in any equilibrium, the flagship store makes no sales.*

*Proof.* Consider an induced steady-state with direct replenishment. $q^o$ being at a steady state requires $q^o(1-\sigma) = \pi(1-\sigma)$. Then, if $q^o$ is at the induced steady state for the shopping strategy $\sigma$, either $q^o = \pi$, or there are no outlet sales $\sigma = 1$.

For the flagship, $q^f$ being induced by a consumer strategy $\sigma$ requires $\sigma q^f = 0$. It follows that in any steady-state induced by $\sigma$ the flagship has no sales: either no consumers shop there, or they can find no high-quality items.

I now show that no interior equilibrium can be supported. Suppose by way of contradiction that consumers shop at both stores with positive probability $\sigma \in (0,1)$. Then, by the above analysis, in any steady state induced by $\sigma$, the outlet has a better product mix: $q^o = \pi > q^o = 0$. However, the consumer strategy is not optimal then: the outlet offers both a quality premium and a discount price.

□

### 2.2.3 Seller's Problem

Theorem 1 highlights the fundamental trade-off faced by the seller: improving product sorting across stores comes at the cost of reducing per-period sales volume. I now explain how the seller's pricing problem in my model differs from the classic problem of setting the optimal monopolistic price.

As shown in Lemma 1, the seller never benefits from exclusively relying on the flagship store ($\sigma = 1$) because it results in poor inventory turnover and ultimately zero sales. Consequently, the seller must either forgo product screening entirely, selling all goods at a low price in the outlet ($\sigma = 0$), or sustain an interior equilibrium where both stores remain active.

The equilibrium relationship between the model's key variables, customer shares across stores, quality composition, total sales, and the flagship price, implies the seller can only raise the flagship price while inducing more consumers to shop there. In particular, Theorem 1 suggests that the seller effectively faces an upward-sloping "demand curve" for her flagship store $\boldsymbol{\sigma}(p^f)$.

Conditional on shopping at the flagship, consumers pay a premium price only if they can find a high-quality item, which happens with probability $\mathbf{q}^f(\sigma(p^f))$. As more consumers go to the flagship, lower-quality items stay unpurchased longer and occupy a larger share of the stock, resulting in fewer per-flagship-shopper transactions. Formally, consider the seller's payoff from an interior equilibrium of the consumers' game for the flagship price $p^f$:

$$
\begin{aligned}
V^S(p^f, \boldsymbol{\sigma}(p^f), \mathbf{q}^f(\boldsymbol{\sigma}(p^f))) &= p^f \boldsymbol{\sigma}(p^f)\mathbf{q}^{\mathbf{f}}(\boldsymbol{\sigma}(p^f)) + v^l(1-\boldsymbol{\sigma}(p^f)) \\
&= \boldsymbol{\sigma}(p^f)(p^f \mathbf{q}^{\mathbf{f}}(\boldsymbol{\sigma}(p^f)) - v^l) + v^l
\end{aligned}
$$

The seller faces a price-setting problem as if she is facing an *upward-sloping* demand $\boldsymbol{\sigma}$ and a *non-linear* per flagship-customer revenue curve $r(p^f) \equiv p^f \mathbf{q^f}(\boldsymbol{\sigma}(p^f)) - v^l$ capturing the benefit of shifting a marginal consumer from outlet to flagship location.

Proposition 2 summarizes the conditions under which the seller prefers to leverage the two-store layout rather than sell all products at low prices: either consumers must value the high-quality good sufficiently how, or the seller is likely enough to produce high-quality goods.

**Proposition 2.** *i) For every $(\pi, v^l)$, there exists $\bar{v}^h(\pi, v^l) \in (v^l, \infty)$ such that the seller strictly benefits (does not benefit) from the two stores if $v^h > (<)\bar{v}^h(\pi, v^l)$.*

*ii) For every $(v^h, v^l)$, there exists $\bar{\pi}(v^h, v^l) \in (0, 1)$ such that the seller strictly benefits (does not benefit) from the two stores if $\pi > (<)\bar{\pi}^h(v^h, v^l)$.*

*Proof.* See Appendix C. □

This proposition is intuitive: the seller can only benefit from a two-store layout when increased high-quality prices can bring enough additional revenue to compensate for a lower total sales volume.

**Frictions of Vertical Integration**. Note that Theorem 1 is true even when the seller is not vertically integrated: it summarizes how the interior equilibrium responds to a change in the flagship price $p^f$, whenever the market abides with the sequential replenishment rule.

In particular, let me parameterize the ease of vertical integration by $\alpha \in (0, 1)$, and assume the seller only earns $\alpha$ share of the outlet revenue. Specifically, one may interpret $(1 - \alpha)v^l$ as the transportation costs associated with moving the items across the locations.

Alternatively, we may assume that the two stores are operated by two different firms, with the flagship seller owning the production plant. The outlet seller purchases the goods from the remaining inventory of the flagship and pays $\alpha v^l$ for every unit of inventory it receives. For this model extension, $\alpha$ can then be interpreted as the bargaining power of the flagship seller.

The consumer's payoff and the low of motion remain the same as in the benchmark model without the frictions of vertical integration. Hence, for every flagship price $p^f$, the interior equilibrium of the consumers' game is characterized by the same function $\boldsymbol{\sigma}, \mathbf{q}^f, \mathbf{q}^o$. But the flagship seller's payoff from any $p^f \in (v^l, v^h)$ now becomes: $V^S(p^f, \boldsymbol{\sigma}(p^f), \mathbf{q}^f(\boldsymbol{\sigma}(p^f))) = \boldsymbol{\sigma}(p^f)p^f\mathbf{q^f}(\boldsymbol{\sigma}(p^f)) + (1 - \boldsymbol{\sigma}(p^f))\alpha v^l$ for some $\alpha \in (0, 1]$. Alternatively, the flagship seller may charge a low price herself and receive a payoff $V^S(v^l, 1, \pi) = v^l$.

**Proposition 3.** *Suppose both stores are operational:* $\sup_{p^f \in (v^l, v^h)} V^S(p^f, \boldsymbol{\sigma}(p^f), \mathbf{q}^f(\boldsymbol{\sigma}(p^f))) > v^l$. *Then, as the frictions for vertical integration go down ($\alpha$ increases), the flagship seller sets a lower price $p^f$ and attracts fewer customers.*

*Proof.* See Appendix C. □

Consequently, consumers benefit from a less frictional vertical integration.

**Corollary 1.** *Suppose both stores are operational. Then, consumers benefit from lower frictions of vertical integration.*

Indeed, by Proposition 3, the seller induces a lower flagship premium as $\alpha$ rises. Then, by Theorem 1, the quality composition at the outlet store increases, and consumers enjoy a higher expected surplus there. In the interior equilibrium, consumers are indifferent between the two stores; hence, they benefit overall.

**Replenishment Frequency and Multiple Stores**. The two-store model in discrete time serves as an intuitive benchmark, offering initial insights into how consumer behavior affects product sorting. Building on these insights, it is natural to ask whether the seller could benefit from operating more stores. What strategies would the seller adopt if she had the flexibility to run as many stores as she wanted? Exploring these questions in discrete time becomes challenging and intractable: When replenishment occurs in discrete time, changes in the quality composition within a store compete with changes introduced by replenishment. This can prevent the quality composition from changing consistently along the replenishment chain.

To resolve this issue, I transition to modeling frequent inventory replenishment in continuous time, an approach used in the subsequent sections of the paper. The reason I presented the two-store model in discrete time first is that, in continuous time, *no product sorting* is achievable when only two stores are operated. This happens because product sorting relies on the difference between the initial and after-sales quality composition at the flagship store. When replenishment occurs continuously, these two compositions coincide, resulting in a degenerate case where both stores have the same quality composition at a steady state: $q^f_a \to q^f$ when $\lambda \to 0$.

# 3 Continuous Model

In this section, I develop a model in continuous time, where the replenishment occurs infinitely often. In this model, the seller is not constrained by any fixed number of stores, operating a continuum of different store locations. The seller selects the joint distribution of prices, consumers, and quality composition, subject to the same steady-state equilibrium constraints as in the two-store model. Despite the richness of the setup, the model remains highly tractable: the seller's problem effectively reduces to choosing a single key parameter—the quality composition at the low-priced locations (Theorem 2). This dimensionality reduction allows for comparative statics analysis and model extensions (Section 4), providing deeper insights into the seller's optimal product sorting.

## 3.1 Model

As in the two-store version, a single long-lived seller serves a continuum of identical short-lived consumers. The seller produces a good whose value to a consumer is uncertain and can be either high $v^h$ or low $v^l$ (where $v^h > v^l > 0$).

**Store Locations.** Building on the two-store framework, this model explores a more flexible and nuanced structure for pricing and product sorting across potentially infinite store locations. To that end, I assume now the seller manages a whole line of *store locations* indexed by $x \in X = (0, 1)$, where index 0 is reserved for the production plant. I normalize the cumulative stock across locations to 1 and assume it is evenly distributed across the available locations. As before, after consumers make their purchasing decisions, the seller replenishes each location to its full capacity.

Given the benefits of sequential replenishment established for the two-store model, I assume the inventory flows in a single direction: inventory is replenished from unsold stock at an immediately preceding location. The direction of product flows is summarized in Figure 4. For the baseline version of the model, I maintain the assumption that the disposal costs are prohibitively high: the seller only depletes her stock through sales (I relax this assumption in Section 4.1).

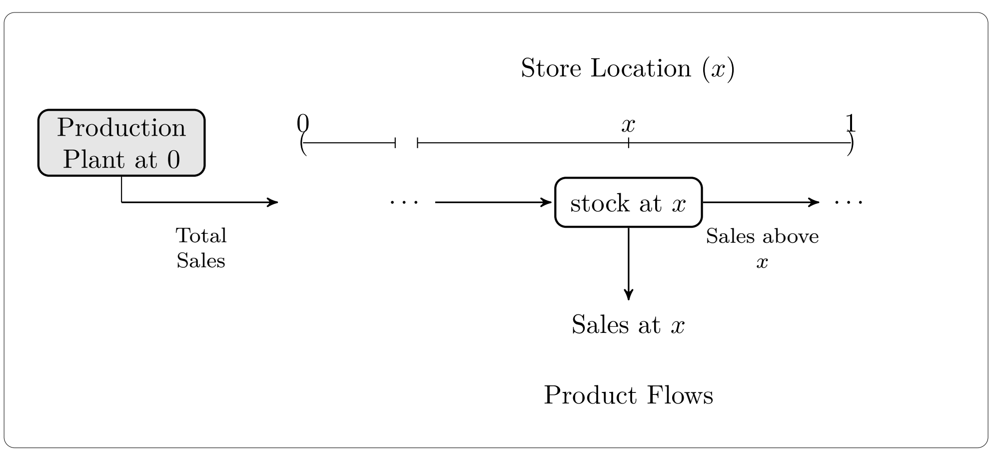



*Figure 4: Linear Inventory Shipments: Product Flows within a Period*

Each store location $x$ is characterized by its price $\mathbf{p}(x)$ and the proportion of high-quality products in its stock $\mathbf{q}(x)$. I assume that both the *price schedule* $\mathbf{p} : X \to \mathbb{R}$ and the *quality composition* $\mathbf{q} : [0, 1) \to [0, 1]$ are (Lebesgue-)measurable. The probability of a high-quality product at the production plant, $\mathbf{q}(0)$, is exogenously given by some $\pi \in (0, 1)$.

I assume that both the prices and the quality composition remain constant over time—i.e., are in *a steady state*. Focusing on the steady state, I keep the model tractable while allowing for an inherently dynamic sorting mechanism across the stores. Despite its advantages, steady-state analysis has its limitations. First, it may exclude richer dynamic strategies by the seller. Second, it provides no insights into the optimal way of reaching the steady state. To address these shortcomings, one would need to consider a more general dynamic model, which lies outside the scope of this paper.

Since there are only two quality levels, the pricing schedule divides store locations into three groups: i) where neither quality is purchased, ii) where only high-value products are purchased, and iii) where both qualities are purchased. I refer to locations in the third group as outlet locations: $x$ is *an outlet location* if $\mathbf{p}(x) \leq v^l$.

**Consumers.** Time is continuous and runs over an infinite horizon. At every instant, a flow of short-lived consumers arrives at the market in a unit mass (i.e., during a time interval of $\Delta$, the mass of arriving shoppers is $\Delta$). This continuous-time approach implies that consumer purchases are small relative to the stock at any location. [11]

Each consumer observes the posted prices, anticipates the steady-state quality composition at every location, and decides where to shop. Let the *consumers' strategy* of a shopping location choice be summarized with a density function $\sigma : X \to \mathbb{R}_+$. [12] As I focus on the steady state, I assume that consumers' strategy is also constant over time.

Consumers' search is random within each store: if location $x$ holds a share $\mathbf{q}(x)$ of high-quality goods, then each buyer has a chance $\mathbf{q}(x)$ of finding a high-quality item at $x$. As before, a buyer breaks ties in favor of the seller when making a purchasing decision[13] and earns a payoff $v^\omega - p$ when purchasing a product of type $\omega \in \{l, h\}$ at a price $p$. The *market outcome* is then fully captured by a tuple of price, consumer shopping strategy, and the steady-state quality composition $\langle \mathbf{p}, \sigma, \mathbf{q} \rangle$.

**Alternative Interpretation: Store Layout**. The model can alternatively be interpreted as the seller managing the layout of a single store. For this interpretation, $X$ summarizes the

[11] That is, the continuous version of the model is a *double* limit of a discrete model—as I both make a period shorter and increase the number of equally-sized stores.

[12] Here, the consumer strategy specifies the mass of consumers shopping over any interval $[x_1, x_2]$ of the store locations: $\int_{y \in (x_1, x_2]} \sigma(y) dy$.

[13] This assumption does not play any substantive role in the analysis since I am only interested in the seller-preferred outcomes.

different locations inside the store, and consumers' shopping strategy is now their allocation of time (or attention) across the different locations within the store. Specifically, we can assume that each consumer has a unit of time they spend at the store, and they decide how much time to spend in either location. Then, each consumer inspects one product at random. The more time (attention) the consumer spends at each location $x$, the more likely he is to pick a product from this part of the store rather than elsewhere. The outlet locations can now be seen as the store's "clearance racks" sections, where the goods are offered at a discount.

**Payoffs**. Both consumers and the seller maximize their flow payoff at the steady state. For a given market outcome, the flow of the consumer surplus is given by:

$$V^B(\mathbf{p}, \sigma, \mathbf{q}) = \int \left[(v^h - \mathbf{p}(x))_+ + (v^l - \mathbf{p}(x))_+\right] \sigma(x) dx$$

and the seller's flow profit is:

$$V^S(\mathbf{p}, \sigma, \mathbf{q}) = \int_{\mathbf{p}(x) \in (v^l, v^h]} \mathbf{p}(x)\mathbf{q}(x)\sigma(x) dx + \int_{\mathbf{p}(x) \leq v^l} \mathbf{p}(x)\sigma(x) dx$$

Let $TS(\cdot)$ denote the total market surplus from any market outcome: $TS(\mathbf{p}, \sigma, \mathbf{q}) = V^S(\mathbf{p}, \sigma, \mathbf{q}) + V^B(\mathbf{p}, \sigma, \mathbf{q})$.

In what follows, I consider how the seller optimally chooses the whole market outcome $\langle \mathbf{p}, \sigma, \mathbf{q} \rangle$ consisting of prices, consumer shopping strategy, and the quality composition at the steady state. The seller faces the same equilibrium constraints as in the two-store model: consumers' behavior and the steady-state quality composition must be mutually consistent given the prices chosen by the seller. I define this formerly in the remainder of the section.

**Induced Steady State**. I start by discussing how the prices and the consumer strategy shape the steady-state quality composition. The products move across the locations for two reasons—shipments due to replenishment and consumer purchases.

Fix some interval $(x_1, x_2]$. The total purchases of high-quality products in the locations within this interval equal the mass of consumers who shop at the locations within $(x_1, x_2]$ having a price (weakly) below $v^h$ and who draw a high-quality product: $\int_{y \in (x_1, x_2], \mathbf{p}(y) \leq v^h} \mathbf{q}(y)\sigma(y) dy$. This is the total mass of high-quality products that move away from the interval $(x_1, x_2]$ due to purchases.

Next, consider the movement in inventory due to shipments. The replenishment policy implies that the total mass of products shipped from any location $x$ is given by the total sales volume in the locations above it—*downstream sales*. The downstream sales are affected by a whole market outcome. In particular, fix a market outcome $m = \langle \mathbf{p}, \sigma, \mathbf{q} \rangle$, then the total

*total downstream sales* $S_m : [0,1] \to \mathbb{R}+$ for location $x$ are given by:

$$S_m(x) = \int_{y>x, \mathbf{p}(y)\in(v^l,v^h]} \mathbf{q}(y)\sigma(y)dy + \int_{y>x,\mathbf{p}(y)\leq v^l} \sigma(y)dy$$

That is, the downstream sales combine the purchases of all product types at the locations above $x$. At the outlet locations, consumers purchase both types of products. At the locations where the prices are within $(v^l, v^h]$, consumers only make a purchase if they find a high-quality product.

Then, the total mass of products that get shipped outside of the interval $(x_1, x_2]$ is given by the downstream sales at $x_2$: $S_m(x_2)$. As time is continuous, there is no within-period quality change due to purchases. Hence, if location $x_2$ holds a share $\mathbf{q}(x_2)$ of high-quality items, then the shipments of high-quality items amount to $\mathbf{q}(x_2)S(x_2)$. Similarly, the mass of high-quality items that get shipped into the interval $(x_1, x_2]$ is given by $S(x_1)\mathbf{q}(x_1)$.

If the quality composition is to remain constant across the locations within the interval $(x_1, x_2]$, it must be that the net effect of inventory shipments $S(x_1)\mathbf{q}(x_1) - S(x_2)\mathbf{q}(x_2)$ must exactly offset all the purchases of high-quality goods made within this interval. Thus, we obtain the following definition for a quality composition being induced by a consumer strategy and prices.

**Definition 1.** For a market outcome $m = \langle \mathbf{p}, \sigma, \mathbf{q}\rangle$, I say that $\mathbf{q}$ is *induced* by $(\sigma, \mathbf{p})$ *on an interval* $[y_1, y_2]$ if for any its subinterval $(x_1, x_2] \subseteq [y_1, y_2]$:

$$\int_{y\in(x_1,x_2],\mathbf{p}(y)\leq v^h} \mathbf{q}(y)\sigma(y)dy = S_m(x_1)\mathbf{q}(x_1) - S_m(x_2)\mathbf{q}(x_2)$$

Similarly, I say that $\mathbf{q}$ is *induced* by $(\sigma, \mathbf{p})$ if the above is true for any interval $[y_1, y_2] \in X$.

Before proceeding further, it is useful to establish the following Lemma 2 that summarizes the main restrictions on the induced steady state for the intervals of non-outlet and outlet locations.

**Lemma 2.** *Consider some market outcome* $m = \langle \mathbf{p}, \sigma, \mathbf{q}\rangle$*.*

*i) Suppose that* $\mathbf{p}(\cdot) > v^l$ *σ-almost surely over* $[x_1, x_2] \subseteq X$*.*[14] *Then,* $\mathbf{q}$ *is induced by* $(\sigma, \mathbf{p})$ *on* $[x_1, x_2]$ *if and only if* $S_m(x)(1 - \mathbf{q}(x))$ *is constant over* $[x_1, x_2]$

*ii) Suppose that* $\mathbf{p}(\cdot) \leq v^l$ *σ-almost surely over* $[x_1, x_2] \subseteq X$ *and* $S_m(x_2) > 0$*. Then,* $\mathbf{q}$ *is induced by* $(\sigma, \mathbf{p})$ *on* $[x_1, x_2]$ *if and only if* $\mathbf{q}(x)$ *is constant over* $[x_1, x_2]$

[14] That is, $\int_{\mathbf{p}(y)\leq v^l, y\in[x_1,x_2]} \sigma(y)dy = 0$. More generally, I say that $A$ holds $\sigma$-almost surely over a set $Y$ if $\int_{y:\neg A, y\in Y} \sigma(y)dy = 0$.

*Proof.* See Appendix G for a proof. □

In words, Lemma 2 states that one of the two things must remain constant over any interval, where the price does not cross $v^l$: the inventory shipments of the low-quality items or the quality composition itself. Intuitively, when the price is too high, the only movement in the low-quality items is due to inventory shipments, and part (i) of the lemma follows. To get the intuition for why the second part is true, note that consumers purchase both product types over an interval. Hence, we should expect no learning by the market across the two neighboring outlet locations.

**Equilibrium in the Consumers' Game**. Given the prices, the consumers' strategy must be optimal at the induced steady state to form an equilibrium.

**Definition 2.** Say that market outcome $\langle \mathbf{p}, \sigma, \mathbf{q} \rangle$ is *an equilibrium in the consumers' game given a price schedule* if

(i) $\mathbf{q}$ is induced by $(\sigma, \mathbf{p})$

(ii) and for every $x$, such that $\sigma(x) > 0$

$$x \in \operatorname*{argmax}_{y \in X} \mathbf{q}(y)(v^h - \mathbf{p}(y))_+ + (1 - \mathbf{q}(y))(v^l - \mathbf{p}(y))_+$$

Let $\mathcal{E}_{\mathbf{p}}$ denote the set of equilibria for a given price schedule $\mathbf{p}$: a market outcome $m' = \langle \mathbf{p}', \sigma', \mathbf{q}' \rangle \in \mathcal{E}_{\mathbf{p}}$ if $\mathbf{p}' = \mathbf{p}$ and $m'$ is an equilibrium in the consumers' game.

**Admissible Price Schedules.** Not every price schedule admits an equilibrium in the consumers' game: $\mathcal{E}_{\mathbf{p}} = \emptyset$ for some prices $\mathbf{p}$. To avoid such pathological cases, I introduce a mild technical constraint on the admissible prices.

To illustrate, suppose that $\mathbf{p}(x) \leq v^l$ for all $x \in X$ and $\mathbf{p}$ is strictly increasing. From Lemma 2, in any steady state, the quality composition remains constant and coincides with the production mean (a.e.) over the support of the consumers' strategy. However, if $\mathbf{p}$ is strictly increasing, then any consumer would benefit by decreasing an index of her shopping location. No equilibrium of the consumers' game exists for such a price schedule. In a discrete model, we would have no problem of this kind because we could construct an equilibrium where all the consumers shop at the location with the lowest price. To replicate this equilibrium, I impose a technical constraint on the seller's price schedule $\mathbf{p}$, requiring that the lowest price is charged at a non-trivial interval of store locations. I then allow the seller to make such an interval arbitrarily small.[15]

[15] That is, for every measurable price schedule $\mathbf{p} : X \to \mathbb{R}_+$, there exists a sequence of admissible price-schedules converging to $\mathbf{p}$ pointwisely.

Formally, say that a price $\mathbf{p}$ schedule is *admissible* if there exists a non-trivial interval such that all locations inside the interval charge the minimal price $\underline{p} = \inf_{x \in X} \mathbf{p}(x)$. Let $\mathcal{A}$ denote the set of admissible price schedules.

For every admissible price schedule, the set of equilibria in the consumers' game is non-empty (see Lemma 9 in Appendix D). I can now formally define the seller's problem: the seller maximizes her flow profit by selecting an admissible price schedule and any equilibrium in the consumers' game associated with it.

$$\sup_{\mathbf{p} \in \mathcal{A}, (\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p}}} V^S(\mathbf{p}, \sigma, \mathbf{q})$$

## 3.2 Equilibrium Product Sorting

In this section, I present the main result of the model with a continuum of stores, characterizing all possible market outcomes that are equilibria in the consumers' game in Theorem 2. I show that the payoffs of the market participants are essentially pinned down by a single parameter—quality composition at the first outlet location. I then use this result to analyze a seller-optimal market outcome in Section 3.3 show that the seller screens product type more aggressively when consumers value high-quality products more or when the average quality at the production plant improves, provided there is not much uncertainty about the product type.

Theorem 2 below establishes that all the market outcomes that can be sustained as an equilibrium of the consumers' game take the following form: there exists some *outlet threshold* $\hat{x} \in [0, 1]$ that divides all the locations that the consumers visit between outlet and non-outlet locations. Formally, I say that a market outcome is a *$\hat{x}$-threshold market outcome* if $\mathbf{p}(\cdot) > v^l$ on $(0, \hat{x})$ ($\sigma$-a.s) and $\mathbf{p}(\cdot) \leq v^l$ on $[\hat{x}, 1]$ ($\sigma$-a.s.). In addition, Theorem 2 the payoffs to both players are effectively pinned down by the quality composition at this outlet threshold.

**Theorem 2.** *Suppose the market outcome $(\mathbf{p}, \sigma, \mathbf{q})$ is an equilibrium in the consumers' game. Then, it is a $\hat{x}$-threshold market outcome for some $\hat{x} \in [0, 1]$. Furthermore:*

*i) If no consumers shop at outlet locations, i.e., $\int_{\hat{x}}^1 \sigma(y)dy = 0$, both players receive zero payoff: $V^S(\mathbf{p}, \sigma, \mathbf{q}) = V^B(\mathbf{p}, \sigma, \mathbf{q}) = 0$.*

*ii) If some consumers shop at outlet locations, i.e., $\int_{\hat{x}}^1 \sigma(y)dy > 0$, the total surplus is determined by the quality composition at the threshold $\hat{x}$:*

$$TS(\mathbf{p}, \sigma, \mathbf{q}) = \frac{\pi v^h + (1 - \pi) v^l}{\ln\left(\frac{\pi}{1-\pi} \frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1 - \pi) + 1}$$

*Moreover, when* $\mathbf{q}(\hat{x}) < \pi$*, the buyers' payoff is* $V^B(\mathbf{p}, \sigma, \mathbf{q}) = \mathbf{q}(\hat{x})(v^h - v^l)$*, and when* $\mathbf{q}(\hat{x}) = \pi$*,* $V^B(\mathbf{p}, \sigma, \mathbf{q}) \geq \mathbf{q}(\hat{x})(v^h - v^l)$*.*

Thus, Theorem 2 effectively summarizes the limitations of the seller's strategies for the continuous version of the model. Despite the seemingly rich space of available seller choice, the equilibrium conditions of the consumers' game imply that the seller only selects the degree to which the high-quality goods are screened out before they reach an outlet threshold.

*Proof.* First, I show that unless the seller makes zero steady-state sales, she must have a positive measure of consumers shopping at outlet locations. Intuitively, if no consumers shop at the outlet locations, low-quality items eventually fill all available shelf space across all store locations. As consumers only want to buy high-quality goods at higher prices, they stop making any purchases when facing only undesirable products.

**Lemma 3.** *Consider any market outcome* $m = \langle \mathbf{p}, \sigma, \mathbf{q} \rangle$ *with* $\mathbf{p} \in \mathcal{A}, \sigma, \mathbf{q} \in \mathcal{E}_{\mathbf{p}}$*. If there are no sales at the outlet locations* $\int_{\mathbf{p}(x) \leq v^l} \sigma(x)dx = 0$*, then the seller makes zero total sales:* $S_m(0) = 0$*.*

*Proof.* Suppose otherwise, $S_m(0) > 0$ and $\int_{\mathbf{p}(x) \leq v^l} \sigma(x)dx = 0$. Then, by Lemma 2 (i), $S_m(x)(1 - \mathbf{q}(x))$ is constant over $X$ and converges to 0, since $S_m(1) = 0$ and $S_m$ is continuous. This is only possible if $S_m(x)(1 - \mathbf{q}(x)) = 0, \forall x \in X$. By Lemma 10 (i) in Appendix F, $S_m(x)(1 - \mathbf{q}(x))$ is right-continuous at 0 if $S_m(0) > 0$, so that we must have $S_m(0)(1 - \mathbf{q}(0)) = 0$. However, $\mathbf{q}(0) = \pi < 1$, and we obtain a contradiction with $S_m(0) > 0$. □

Suppose no consumers shop at the outlet locations under some market outcome $m = \langle \mathbf{p}, \sigma, \mathbf{q} \rangle$. Then, $m$ is a threshold market outcome for a threshold outlet $\hat{x} = 1$. By Lemma 3, the seller makes zero steady-state sales in such a market outcome, and hence the total surplus is also 0. This delivers part (i) of the additional result in the theorem.

On the other hand, when a positive measure of consumers shop at the outlet locations, the steady-state sales are strictly positive, as consumers purchase both types of products at such locations. Then, due to Lemma 3, we obtain the sales are positive if and only if a positive measure of consumers shop at the outlet locations.

Let us restrict attention to the market outcomes with positive sales from now on. Lemma below confirms that all such market outcome are also threshold market outcomes.

**Lemma 4.** *Suppose a market outcome* $(\mathbf{p}, \sigma, \mathbf{q})$ *is an equilibrium in the consumers' game. Then,* $(\mathbf{p}, \sigma, \mathbf{q})$ *is a threshold market outcome for the outlet threshold* $\hat{x} = \inf\{x \in X : \mathbf{p}(x) \leq v^l\}$*.*

*Proof.* See Appendix G. □

Intuitively, Lemma 4 holds because the quality composition is non-increasing (due to Lemma 2 (i)). When prices are high, due to consumers skimming off high-quality goods and the quality of the products down, the replenishment chain gets worse. Because of this, consumers will never have an incentive to shop at the locations that charge a high price past a threshold $\hat{x}$—they offer a worse product mix and higher prices. As a result, the locations naturally divide into two groups: one where consumers buy only high-quality products at higher prices and another where consumers buy any product quality at lower outlet prices.

For threshold mechanisms, we can characterize an induced steady state using Lemma 2. In particular, on the interval containing non-outlet locations $(0, \hat{x})$, the evolution of the relative likelihood between the two quality levels is captured by the Lambert function $W : \mathbb{R}_{++} \to \mathbb{R}_{+}$, where $W(x)$ is implicitly defined as:

$$W(x)e^{W(x)} = x$$

Lemma 5 highlights the interdependence between the consumers' selection of stores and the speed at which the quality composition decreases along the replenishment chain.

**Lemma 5.** *For any threshold market outcome $\langle \mathbf{p}, \sigma, \mathbf{q} \rangle$ with an outlet threshold $\hat{x}$ and positive sales, $\mathbf{q}$ is induced by $(\sigma, \mathbf{p})$ on $[0, \hat{x}]$ if and only if for every $x \in [0, \hat{x}]$:*

$$\frac{\mathbf{q}(x)}{1 - \mathbf{q}(x)} = W\left(\frac{\pi}{1-\pi} \exp\left[\frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{(1 - \mathbf{q}(\hat{x})) \int_{\hat{x}}^1 \sigma(y)dy}\right]\right)$$

$$\text{for } (1 - \mathbf{q}(\hat{x})) \left(\int_{\hat{x}}^1 \sigma(y)dy\right) \left[\ln\left(\frac{\pi}{1-\pi}\frac{1 - \mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right) + \frac{1}{1-\pi}\right] = 1 \tag{Q-T}$$

*Proof.* See Appendix G. □

Equipped with Lemma 5, I can now characterize the total surplus of any market outcome with positive sales and show that it only depends on the outlet threshold's quality composition.

**Lemma 6.** *In any threshold market outcome $\mathbf{p} \in \mathcal{A}, (\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p}}$ with an outlet threshold $\hat{x}$ and positive sales:*

$$TS(\mathbf{p}, \sigma, \mathbf{q}) = \left(\pi v^h + (1-\pi)v^l\right) / \left(\ln\left(\frac{\pi}{1-\pi}\frac{1 - \mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1-\pi) + 1\right)$$

*with $\mathbf{q}(\hat{x}) \in (0, \pi]$. In addition, $V^B(\mathbf{p}, \sigma, \mathbf{q}) = \mathbf{q}(\hat{x})(v^h - v^l)$ if $\mathbf{q}(\hat{x}) < \pi$ and $V^B(\mathbf{p}, \sigma, \mathbf{q}) \geq \mathbf{q}(\hat{x})(v^h - v^l)$ if $\mathbf{q}(\hat{x}) = \pi$.*

*Proof.* By definition, locations $[\hat{x}, 1)$ are outlet locations $\sigma$-a.s. Hence, for every location in $x \in [\hat{x}, 1)$, the total downstream sales in a market outcome $m = \langle \mathbf{p}, \sigma, \mathbf{q} \rangle$ coincide with the total mass of consumers shopping above $x$: $S_m(x) = \int_x^1 \sigma(y)dy$. Clearly, $S_m(\cdot) > 0$ for all locations in $[\hat{x}, 1)$ ($\sigma$-a.s.). Then, from Lemma 2 (ii), ($\sigma$-a.s.) all of them have the same quality composition, equal to $\mathbf{q}(\hat{x})$.

Hence, the cumulative total surplus from locations above $\hat{x}$ is

$$\left[\mathbf{q}(\hat{x})v^h + (1 - \mathbf{q}(\hat{x}))v^l\right] \int_{\hat{x}}^1 \sigma(y)dy$$

—the expected quality of the good times the number of consumers shopping at these locations. As a positive share of consumers shop in locations in $[\hat{x}, 1)$, which have the same quality composition $\mathbf{q}(\hat{x})$ and charge a price of at most $v^l$-$\sigma$-a.s., consumer payoff is at least $\mathbf{q}(\hat{x})(v^h - v^l)$.

By Equation (Q-T) in Lemma 5, $\mathbf{q}(\hat{x}) \in (0, \pi]$ whenever $\int_{\hat{x}}^1 \sigma(y)dy > 0$, which is true as long as the market outcome has positive sales due to Lemma 3.

The price is low at $\hat{x}$ (or its right neighborhood) and therefore offers a positive consumer surplus ($\mathbf{q}$ is continuous at $\hat{x}$ by Lemma 10 in Appendix F).

Consequently, consumers must at least purchase high-quality items at all locations they visit. As all preceding locations $x < \hat{x}$ are non-outlet locations, then they only sell high-quality items. Then, the total surplus from locations $(0, \hat{x})$ in a market outcome $m = \langle \mathbf{p}, \sigma, \mathbf{q} \rangle$ equals the total sales on this interval times the value of the high-quality product $v^h \left[S_m(0) - S_m(\hat{x})\right] v^h$.

As prices are high at all locations on $(0, \hat{x}]$, the inventory shipments of low-quality products must exactly offset each other at the induced steady state on $[0, \tilde{x}]$: $S_m(0)(1-\pi) = S_m(\hat{x})(1-\mathbf{q}(\hat{x}))$ by Lemma 2.

Recall that total downstream sales at $\tilde{x}$ coincide with the mass of consumers shopping above it: $S_m(\hat{x}) = \int_{\hat{x}}^1 \sigma(y)dy$—as locations $[\hat{x}, 1)$ are outlet $\sigma$-a.s. Then, we obtain:

$$S_m(0)(1 - \pi) = S_m(\hat{x})(1 - \mathbf{q}(\hat{x})) = (1 - \mathbf{q}(\hat{x})) \int_{\hat{x}}^1 \sigma(y)dy$$

and the total surplus simplifies as follows:

$$\begin{aligned} TS(\mathbf{p}, \sigma, \mathbf{q}) &= v^h \left[S_m(0) - S_m(\hat{x})\right] + \left[\mathbf{q}(\hat{x})v^h + (1 - \mathbf{q}(\hat{x}))v^l\right] \int_{\hat{x}}^1 \sigma(y)dy \\ &= v^h \frac{\pi - \mathbf{q}(\hat{x})}{1 - \pi} \int_{\hat{x}}^1 \sigma(y)dy + \left[\mathbf{q}(\hat{x})v^h + (1 - \mathbf{q}(\hat{x}))v^l\right] \int_{\hat{x}}^1 \sigma(y)dy \end{aligned}$$

$$= \left(\pi v^h + (1-\pi)v^l\right) \frac{1-\mathbf{q}(\hat{x})}{1-\pi} \int_{\hat{x}}^{1} \sigma(y)dy$$
$$= \left(\pi v^h + (1-\pi)v^l\right) / \left(\ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1-\pi)+1\right)$$

where the last equality follows from Equation (Q-T) in Lemma 5.

Finally, note that whenever $\mathbf{q}(\hat{x}) < \pi$, a positive measure of consumers shop at non-outlet locations (due to Equation (Q-T) in Lemma 5). In this case, I show in Lemma 13 in Appendix F that consumers shop at prices (weakly) above $v^l$ $\sigma-$a.s. Then, it follows that the consumer payoff is pinned down by $\mathbf{q}(\hat{x})$: $V^B(\mathbf{p}, \sigma, \mathbf{q}) = \mathbf{q}(\hat{x})(v^h - v^l)$.

Intuitively, this is because the quality composition changes continuously at $\hat{x}$, and if the price was to get below $v^l$, the consumers would prefer to deviate to such outlet locations, as they offer only a marginally worse quality composition but a discretely better price. □

Lemma 6 implies part (ii) of the additional part of the theorem, which completes the proof. □

Theorem 2 characterizes which payoffs can ever arise in some equilibrium in the consumers' game. In a complementary result of Proposition 4, I show that the seller can always construct a market outcome that would sustain any payoff as in Theorem 2 (ii).

**Proposition 4.** *For every $q \in (0, \pi]$, there exists a $\hat{x}$-threshold market outcome $\mathbf{p} \in \mathcal{A}, (\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p}}$, such that:*

*i) it has a positive measure of outlet shoppers*

*ii) the outlet-threshold $\hat{x}$ has the quality composition $q$: $\mathbf{q}(\hat{x}) = q$*

*iii) and attains a lower boundary of consumer payoff $V^B(\mathbf{p}, \sigma, \mathbf{q}) = q(v^h - v^l)$*

*Proof.* See Appendix G. □

In Appendix G, I construct such a market outcome for every $q$ with a uniform consumer strategy $\sigma(x) = 1$. To construct the quality composition, I rely on Lemma 5 and Lemma 2. The prices are then pinned down by the consumers' indifference condition across the locations.

## 3.3 Seller-Optimal Product Sorting

In this section, I analyze the main properties of a seller-optimal market outcome, focusing on how its shape is affected by the model's parameters. In particular, I show that as high-quality goods become more valuable or prevalent, optimal product sorting becomes more profitable to the seller. In addition, as the value of the high-quality product increases, the seller aims for a more aggressive screening between the different locations to charge higher prices.

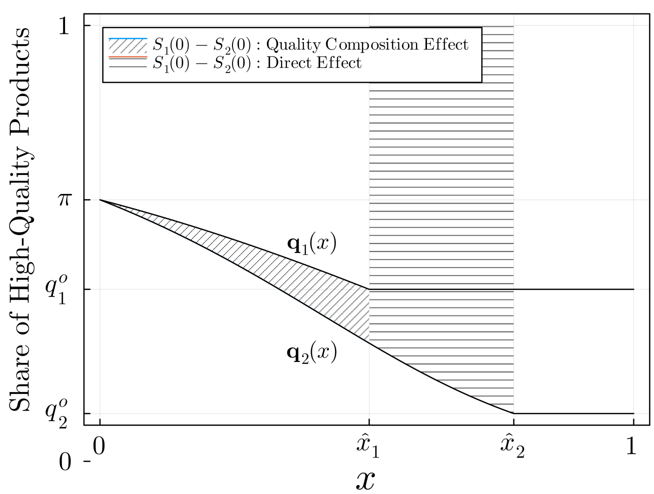
Given Theorem 2 and Proposition 4, we can reduce the seller's problem to the choice of a quality composition at the threshold outlet location.

**Corollary 2.** *Suppose that a $\hat{x}$-threshold market outcome $\mathbf{p} \in \mathcal{A}$, $(\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p}}$, then the quality composition at the outlet threshold $\mathbf{q}(\hat{x}) \in \underset{q \in (0,\pi]}{Argmax}\, \tilde{V}^S$ for:*

$$\tilde{V}^S(q) = \frac{\pi v^h + (1-\pi)v^l}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)(1-\pi)+1} - q(v^h - v^l)$$

**Sorting/Sales Trade-Off.** Note that in the continuous version of the model, the seller faces the same trade-off between sales volume and the degree of product sorting. As the seller aims to screen the products more aggressively and leave fewer high-quality goods for the outlet locations, more buyers must shop at high prices. In turn, this exacerbates adverse selection for the stock at all locations, and the quality composition worsens everywhere. Consequently, the market outcome becomes less efficient: the total sales volume declines as more consumers purchase only one type of product upon inspection (direct effect), and finding a high-quality good becomes harder (quality composition effect).

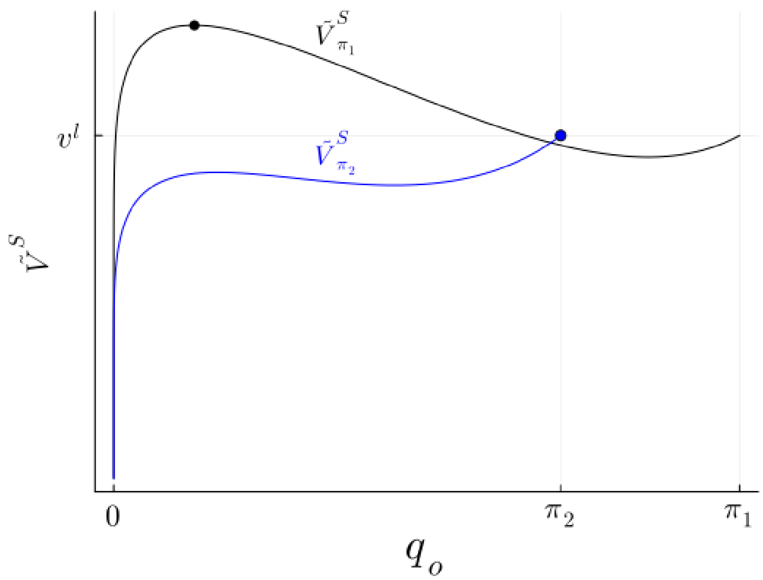
*Figure 5: Steady-State Quality Composition*




*Note*: the figure depicts the effect of a decrease in the targeted outlet quality composition ($q_1^o > q_2^o$) on the steady-state quality distribution and sales volume. $\mathbf{q}_i(\cdot)$ and $S_i(0)$ denote the steady-state quality composition and total sales volume given the outlet quality $q_i^o$. The hashed area plots the associated sales loss as the seller screens products more aggressively, decomposing it into a direct effect (west to east hatch) and the quality composition effect (northeast to southwest hatch).

From Corollary 2, the seller only chooses the quality composition at the outlet threshold locations, which is the main focus of the comparative statics exercises in this section. First, in Appendix H, I establish an analog of Proposition 2 for the continuous model: The seller offers outlet locations only and does no product sorting when either the consumer's valuation of high-quality goods ($v^h$) or the probability of producing high-quality goods ($\pi$) is low (Proposition 9).

*Figure 6: Seller's Payoff*




*Note*: the figure depicts the seller's payoff as a function of the outlet threshold quality composition for different proportions of the high-quality product at the production plant, with $\pi_2 > \pi_1$. The dots of the respective color are plotted at the seller's optimum.

With a continuous model, I can do a more precise comparative statics exercise studying how the optimal screening intensity changes with the model parameters. As $v^h$ increases, the intensity of screening rises—the quality composition at the outlet threshold locations decreases. Consequently, the seller can charge higher prices at all non-outlet locations. Similarly, the seller sorts products more aggressively when the production technology improves ($\pi$ rises), provided there is little uncertainty in the product quality. I summarize these comparative statics exercises in Proposition 5.

**Proposition 5.** *The optimal quality composition at the outlet locations decreases if*

*i)* $v^h$ *goes up,*

*ii) or* $\pi \approx 1$ *and increases.*

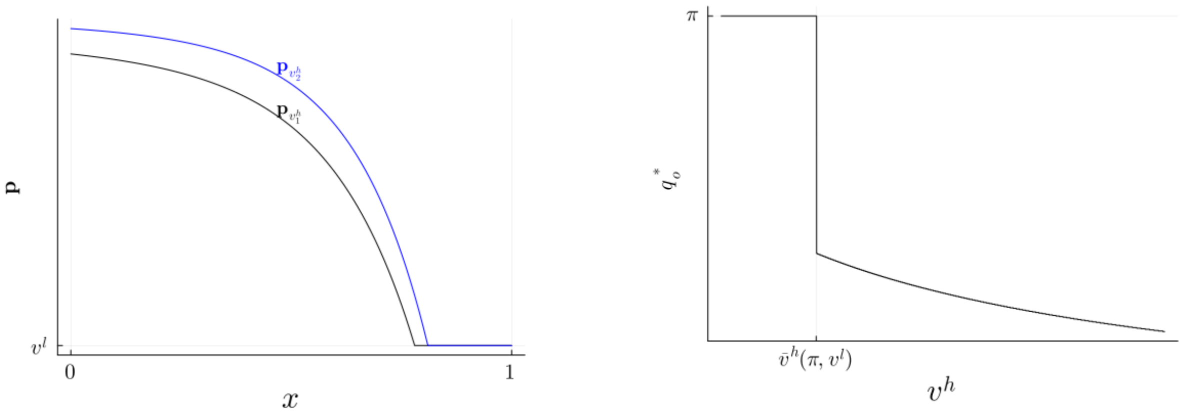
*In addition, as* $v^h$ *increases, the price increases at every location in* $X$ *for the seller-optimal with a uniform consumer strategy.*

*Proof.* See Appendix H. □

*Figure 7*

*(a) Optimal Price Schedule*

*(b) Optimal Quality Composition at Outlet Locations*




*Note*: the figure illustrates that the price rises for all locations at an optimal price schedule as the value of a high-quality good increases from $v_1^h$ to $v_2^h$.

*Note*: the figure illustrates that the optimal screening intensity rises with the value of a high-quality good. Specifically, it illustrates that the quality composition at the outlet threshold worsens as $v^h$ increases.

Consequently, Proposition 5 (ii) implies that the consumer's payoff is non-monotone with respect to $\pi$.

**Corollary 3.** *The consumer's payoff at a seller's optimal market outcome is non-monotone with respect to* $\pi$*.*

Intuitively, on the one hand, a higher probability of high-quality products makes the production technology more favorable and potentially presents a greater chance of encountering a high-quality good. The consumers enjoy this positive effect fully when $\pi \approx 0$ and the seller does no product sorting. On the other hand, a higher $\pi$ decreases the information asymmetry between the consumers and the seller. Consequently, the seller may extract a higher consumer surplus with lower efficiency losses.

# 4 Discussion

In this section, I discuss the limitations of my model. Some of these limitations I can address with appropriate extensions. First, I relax the assumption of prohibitively high replacement costs and allow the seller to dispose of unsold inventory directly. I show that the seller uses only one disposal channel and either offers outlet stores or disposes of unsold stock directly, keeping prices high. I also generalize this insight for the model with multiple quality tiers.

Second, I relax the technical restriction on the distribution of consumers across the stores by allowing it to admit atoms. I show that the seller benefits from having an unlimited number of non-outlet locations, which allows her to make the product screening more precise. Last but not least, I formulate the extension of the benchmark model that allows for heterogeneous consumers. I show that a positive sorting arises between the consumer types and the quality composition at their shopping location.

I also highlight the questions that this paper does not address. Specifically, I explain why allowing depreciation of the product quality makes the model less tractable. In addition, I point out the potential future direction of the research that would account for the product flow design across multiple location stores.

For brevity, I do not provide a full exposition of the model extensions and focus on its main insights. For the complete formulation, see Appendix E (for the models considered in Section 4.1 - Section 4.3) and Appendix L (for the model of Section 4.4).

## 4.1 Direct Disposal

As a first extension, I allow for the direct disposal of unsold items. Proposition 6 establishes that the seller uses only one channel when disposing of low-quality products. The seller offers outlet locations if the replacement/disposal cost is sufficiently high. Otherwise, the seller offers only disposes of the low-quality products directly by destroying them.

Now, I consider a version of the model that allows for the direct disposal of the products. In particular, I assume the seller may also choose to destroy the unsold inventory at a rate $\gamma$ at some per-unit disposal cost $\kappa > 0$. Equivalently, $\kappa$ can be interpreted as a per-unit production cost. For this interpretation of the model, the value of the product of type $v^{\omega}$ is the consumer's value for the product *net* of the product costs.

To preserve the linear nature of the inventory shipments, the products that get sent away are stored at the location with an index of 1 (for technical simplicity, location 1 is not available for shopping and is only used for storing the products that are to be sent away). Note that we can keep the model the same but must allow this special location 1 to make sales at a negative price of $-\kappa$. In Proposition 6, I summarize the main result for this version of the model, stating the seller uses only one disposal method.

**Proposition 6.** *The seller disposes of the low-quality goods through one channel only. In particular, there exists* $\bar{\kappa}(v^h, v^l, \pi) \in \left( \max_{q \in (0,\pi]} \tilde{V}^S(q) - v^l, \frac{\pi}{1-\pi} v^h \right)$*, such that*

*i) if* $\kappa < \bar{\kappa}(v^h, v^l, \pi)$*, then in any optimal market outcome, there are no outlet locations* $\mathbf{p}(x) > v^l$

*ii) if $\kappa > \bar{\kappa}(v^h, v^l, \pi)$, then there is no direct disposal $\gamma = 0$*

*In addition, $\bar{\kappa}(v^h, v^l, \pi)$ is decreasing in the value of a low-quality product $v^l$.*

*Proof.* See Appendix I. □

The seller faces a trade-off between relying on consumers to buy low-quality goods at discounted prices or paying a cost to dispose of them. If the cost of disposal is relatively low, the seller finds it cheaper to clear shelf space quickly by destroying the product herself rather than offering discounts to attract outlet shoppers. Conversely, if disposal costs are high, the seller prefers to incentivize consumers to purchase low-quality products.

## 4.2 Atoms in the Shopping Strategy

In this section, I extend Theorem 2 to a more general shopping strategy. The total surplus only depends on the same mass of outlet shoppers and the quality composition at the outlet threshold. Using this technical result, I then show that any market outcome with finitely many non-outlet locations is seller-suboptimal. Consequently, we may conclude that the seller benefits from an infinite number of stores as it makes the product screening more precise.

Suppose the shopping strategy is summarized by a cdf $D : [0, 1] \to [0, 1]$ where $D(x)$ denotes the total mass of consumers shopping in $(0, x]$. Now, the shopping strategy $D$ may potentially be discontinuous, allowing for atoms in consumer distribution across the locations in $(0, 1)$. It turns out the seller finds it suboptimal to make the customer share of any non-outlet location large.

**Theorem 3.** *Any market outcome $(\mathbf{p}, D, \mathbf{q}, \gamma)$ that is an equilibrium in the consumers' game is a $\hat{x}$-threshold market outcome for some $\hat{x} \in [0, 1]$. Furthermore, if the total sales are positive, the total market surplus is given by*

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) = \left( \int_{\hat{x}}^{1} dD(y) + \gamma \right) (1 - \mathbf{q}(\hat{x}-)) \left( \frac{\pi}{1 - \pi} v^h + v^l \right) - \gamma \mathbf{q}(\hat{x}-)(v^h - v^l) - \gamma(\kappa + v^l)$$

*In addition, if $D$ admits finitely many discontinuities at non-outlet locations, then the market outcome $\mathbf{p} \in \mathcal{A}, (D, \mathbf{q}) \in \mathcal{E}_{\mathbf{p},\gamma}$ is suboptimal for the seller.*

*Proof.* See Appendix G. □

The first part is simply an extension of Theorem 2. The main proof steps are the same. Note that for the same share of outlet shoppers, the seller wants to do as much screening as possible. Indeed, due to Theorem 2, the total surplus is decreasing in $\mathbf{q}(\hat{x}-)$. In addition, more aggressive product screening allows for better consumer surplus extraction.

The intuition of the additional part of the theorem can be formulated as follows. If an atom is at some non-outlet location, then the seller's learning is "bunched," and she wastes some of the sorting opportunities. The seller could benefit by making the learning slower between the locations, as it makes her own selection of products used for inventory replenishment more fine-tuned. Essentially, the seller uses consumer purchasing decisions to sift out lower-quality products. By having a larger number of store locations, the seller makes the sift denser, which allows for more efficient separation of high-quality items. Lemma 15 in Appendix F provides a formal proof of this argument.

This extension also highlights a key distinction between the model presented here and traditional monopolistic screening models. With two consumer types, the seller only needs two menus to differentiate between the different consumer groups. In contrast, when it comes to sorting different product types, the seller prefers to have infinitely many menus (locations), even when the product types themselves are also binary.

## 4.3 Multiple Quality Tiers

In this section, I demonstrate how the model can be generalized to accommodate richer quality differences of the products. Proposition 7 establishes that the key insights continue to hold even when the product quality is non-binary: the seller's problem is still one-dimensional and can be reduced to the choice of the consumer surplus.

Suppose now that the product has $N$ quality levels, which bring consumer values $v^1 > v^2 > \cdots > v^N$, respectively. I revert to a simpler version of the model where the shopping strategy admits a density. The steady-state quality composition is now described by $\mathbf{q} : \{1, \ldots, N\} \times [0,1] \to [0,1]$, where $\mathbf{q}(i|x)$ denotes the share of quality $i$ at location $x$.

**Proposition 7.** *Consider any two market outcomes $\langle \mathbf{p}, \sigma, \mathbf{q}, \gamma \rangle$ and $\langle \mathbf{p}', \sigma', \mathbf{q}', \gamma \rangle$. Then,*

*i)* $V^S(\mathbf{p}, \sigma, \mathbf{q}, \gamma) = V^S(\mathbf{p}', \sigma', \mathbf{q}', \gamma)$ *whenever* $V^B(\mathbf{p}, \sigma, \mathbf{q}, \gamma) = V^B(\mathbf{p}', \sigma', \mathbf{q}', \gamma)$

*ii) If $\gamma = 0$, then in any market outcome with positive sales, there is a positive measure of buyers shopping at prices no higher than $v^N$: $\int_{\mathbf{p}(y) \le v^N} \sigma(y)dy > 0$.*

*iii) If $\gamma > 0$, there exists $\underline{p}$ such that if $v_i < \underline{p}$, quality $i$ is only directly disposed of.*

*Proof.* See Appendix K. □

The first part of Proposition 7 indicates that the induced payoff of the buyer essentially pins down the seller's market outcome. The key steps of the proof rely on the following two observations. First, the quality composition can change between the different store locations only due to the consumer making different product purchasing decisions. As such, unless the price crosses $\{v^i\}_{i=1}^n$, the relative composition of products *conditional on the purchasing decision* remains constant (see Lemma 11 (ii) in Appendix F).

Second, to satisfy the indifference condition for the consumers, the change in the quality composition must exactly compensate for the price change between any two locations in the (interior) of the support of $\sigma$. The first observation above also implies that the expected value of a product conditional on purchase also remains constant unless the price crosses one of the product's values. Hence, whenever the price changes from $v_i$ to $v_{i+1}$ on some interval, the purchase probability must change just enough to leave the consumer indifferent throughout the interval.

Generalizing Lemma 4, I show that i) in every market outcome, the price crosses each of $\{v^i\}_{i=1}^n$ at most once, inducing a partition over $X$; and ii) the targeted change in purchasing probability determines the mass of consumers who shop at prices between any $v_i$ and $v_{i+1}$. As a result, the role of price is still very limited, even with multiple quality levels. The seller effectively only needs to set two threshold prices: the highest price (through the selection of the consumer surplus) and the lowest threshold price (by appropriately adjusting the disposal rate).

## 4.4 Consumer Heterogeneity

In this section, I formulate the main extension of the benchmark model that allows for heterogeneous consumers. I show that a positive sorting arises between the consumer types and quality composition at their shopping location. Specifically, higher consumer types shop at earlier locations. The outlet locations are used both for product assortment management and consumer segmentation. The seller's problem can be reduced to choosing the highest type of shopping at an outlet location.

Consider the model with a binary quality from Section 3, but assume that the consumers are heterogenous in their valuation of a high-quality product. Specifically, I assume that a consumer of type $\theta$ values a high-quality product at $\theta \in \Theta = [v^l, v^h]$ and values a product of low quality at $v^l$.[16] I assume that $F : \Theta \to [0, 1]$ is a cdf over the possible consumer types, which admits a density $f : \Theta \to \mathbb{R}$ over the entire support $\Theta$.

As before, each shopping location in $X$ is characterized by its price and quality composition, which are observable to all consumers. Consumers of each type select a shopping

[16]For interpretation, I assume that $v^l$ represents a utilitarian value of the product, whereas $\theta$ captures a taste for fashion.

location, and, for simplicity, I focus on the market outcomes that fully separate the different types of consumers. That is, I assume that for any market outcome, an injective function $\mathbf{x} : \Theta \to X$ exists, such that all consumers of type $\theta$ shop at location $\mathbf{x}(\theta)$.

The payoffs of the agents now become:

$$V^B(\mathbf{p}, \mathbf{x}, \mathbf{q}|\theta) = \mathbf{q}(\mathbf{x}(\theta))(\theta - \mathbf{p}(\mathbf{x}(\theta))_+ + (1 - \mathbf{q}(\mathbf{x}(\theta)))(v^l - \mathbf{p}(\mathbf{x}(\theta))_+$$
$$V^S(\mathbf{p}, \mathbf{x}, \mathbf{q}) = \int_{\mathbf{p}(\mathbf{x}(\theta)) \leq \theta} \mathbf{p}(\mathbf{x}(\theta))\mathbf{q}(\mathbf{x}(\theta))f(\theta)d\theta + \int_{\mathbf{p}(\mathbf{x}(\theta)) \leq v^l} \mathbf{p}(\mathbf{x}(\theta))f(\theta)d\theta$$

As consumers are heterogeneous, the optimality of the consumer's shopping strategy $\mathbf{x}$ now requires that neither of the consumer types has a profitable deviation from the prescribed shopping location to any other store (given the distribution of prices and steady-state quality composition).

**Definition 3.** Say that $\mathbf{q}, \mathbf{x}$ is *an equilibrium in the consumers' game (with heterogeneous types)* if $\mathbf{q}$ is induced by $\mathbf{x}, \mathbf{p}$ and for every $\theta \in \Theta$ and every $x \in X$

$$V^B(\mathbf{p}, \mathbf{x}, \mathbf{q}|\theta) \geq \mathbf{q}(x)(\theta - \mathbf{p}(x))_+ + (1 - \mathbf{q}(x))(v^l - \mathbf{p}(x))_+ \tag{IC}$$

To state the main results of this extended model, it is useful to introduce the induced allocation of quality $Q : \Theta \to [0, 1]$ with $Q(\theta) = \mathbf{q}(\mathbf{x}(\theta))$ for every $\theta$. That is, $Q$ simply summarizes the quality composition encountered by every type in the market outcome.

Proposition 8 establishes negative sorting between consumer types and store locations. Intuitively, higher types must be sorted into locations that offer higher expected quality in any equilibrium. From our earlier analysis, the quality composition is decreasing along the replenishment order, encouraging consumer types to also sort in a decreasing order.

**Proposition 8.** *For every market outcome with positive sales, there exists $\bar{\theta} \in (v^l, v^h]$ such that all types $(\bar{\theta}, v^h]$ shop at non-outlet locations; and types $[v^l, \bar{\theta})$ shop at outlet locations. In addition, $\mathbf{x}$ is decreasing on $[\bar{\theta}, v^h]$.*

*Proof.* See Appendix L. □

Given Proposition 8 and our analysis of a homogenous buyer, we can now characterize the induced allocation of quality. Indeed, the size of the outlet share is now captured by the mass of consumers whose type is below the threshold $\bar{\theta}$. Similarly, as the quality composition is non-increasing, higher types must then choose to shop at earlier locations. Hence, the amount of screening for type $\theta$'s induced quality composition is determined by a mass of consumers of higher types. For a given threshold type $\bar{\theta}$, the induced allocation must then satisfy:

$$\frac{Q^{\bar{\theta}}(\theta)}{1-Q^{\bar{\theta}}(\theta)} = W\left(\frac{\pi}{1-\pi}\exp\left[\frac{\pi}{1-\pi} - \frac{1-F(\theta)}{F\left(\bar{\theta}\right)\left(1-Q^{\bar{\theta}}\left(\bar{\theta}\right)\right)}\right]\right)$$

and $Q^{\bar{\theta}}(\bar{\theta})$ is satisfies:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-Q^{\bar{\theta}}(\bar{\theta})}{Q^{\bar{\theta}}(\bar{\theta})}\right) = \frac{1}{F\left(\bar{\theta}\right)\left(1-Q^{\bar{\theta}}\left(\bar{\theta}\right)\right)} - \frac{1}{1-\pi}$$

The standard argument (as in Mussa and Rosen (1978)) can be used to establish the payoff for every consumer type shopping at non-outlet locations satisfies the standard envelope condition. Consequently, a consumer's virtual type determines the seller's additional payoff (relative to charging $v^l$ at all stores) at non-outlet locations (conditional on purchase). Contrary to Mussa and Rosen (1978), conditional on the market segmentation between outlet and non-outlet locations, the seller cannot decide on the offered menus and relies on the consumers to form the menus for her by sorting the product types through sales. Consequently, the seller's problem reduces to segmenting her customer base between the outlet and non-outlet locations by selecting the threshold type $\bar{\theta}$. I summarize this observation in a corollary below.

**Corollary 4.** *In a model with heterogenous consumers, the seller's problem is equivalent to the optimal choice of $\bar{\theta} \in (v^l, v^h]$ so that to maximize*

$$\int_{\bar{\theta}}^{v^h} Q^{\bar{\theta}}(\theta)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)d\theta - (1-F(\bar{\theta}))Q^{\bar{\theta}}\left(\bar{\theta}\right) + v^l$$

## 4.5 Unaddressed Model Limitations

In this section, I highlight key questions that fall outside the scope of this paper but offer promising avenues for future research.

**Quality Depreciation.** So far, I have assumed that the preferences for any particular product remain constant over time. But in real life, even popular designs lose customer appeal with time. For instance, in the apparel industry, this may happen due to the seasonality of products. Within this paper, one could accommodate time depreciation by assuming that, with some probability, a unit of unsold high-quality inventory loses its value and becomes of low quality.

In Appendix A, I extend the two-store model to verify that the model's main market outcomes remain robust in the presence of depreciation. Specifically, the same co-movement between the model's key variables remains true when allowing for depreciation. That is, the same trade-off between sorting and the sales volume is still present.

However, with time depreciation, the irrelevance result for the continuous model no longer holds: the quality at the outlet locations does not simply depend on the "screening budget" (which is pinned down by the mass of non-outlet shoppers) but also depends on the average time the goods spend unsold before reaching the outlet. Consequently, the seller gets a new leverage of exploiting time and must balance a new trade-off. The seller can "speed up" turnover and dampen the effect of depreciation by increasing the customer share of the earlier locations. By doing so, the seller improves the average quality composition and increases sales volume at high-priced locations but prevents the goods from getting damaged before reaching outlets.

**More General Product Flows.** My model has greatly constrained the seller's usage of sales performance data. As argued in the introduction, this assumption may be justified by the high cost of more nuanced inventory management decisions due to the great volumes of inventory items. Nevertheless, even lacking the ability to track each item's performance individually, it seems reasonable to explore how the seller can use other automated inventory shipment rules (not necessarily linearly ordered) to redistribute the unsold stock across the different locations.

For instance, a natural question to ask is whether the seller would benefit from having two separate lines or brands, both with their high-quality stores and own outlets. Similarly, if the seller manages a single outlet line for the two brands, would she benefit from merging them? Given the model's tractability, it seems promising to allow for the analysis of these richer shipment rules within the suggested framework.

**Frequency of Replenishment.** As argued before for the two-store model, the frequency of inventory replenishment offers another strategic tool for the seller to enhance product sorting efficiency. Exploring the impact of replenishment frequency, particularly in scenarios where stock-outs occur, could provide additional valuable insights.

I believe these potential model generalizations will provide a more comprehensive understanding of optimal product sorting and offer new insights into how sellers navigate consumer behavior, pricing, and inventory management in increasingly complex market environments.

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# Appendices for the Two-Store Model

## A Two-Store Model with Depreciation

In this Appendix, I consider a more general two-store model allowing for time depreciation. Suppose that after consumers make their purchases but before the shipments are made, a high-quality unit may lose its value with probability $\delta$. Consider the sequential replenishment. The total outflow of the high-quality goods now consists of consumer purchases $q^f\sigma\lambda$, depreciation of remaining high-quality goods $\delta q^f(1-\lambda\sigma)$, and the shipments to the outlet $\frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\sigma\lambda}(1-\sigma)$. Here, $\frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\sigma\lambda}$ is the after-sales and after-depreciation proportion of high-quality items at the flagship. Equating the total outflow to the inflow from the production, the flagship quality $q^f$ is induced by $\sigma$ and $p^f \in (v^l, v^h)$, $p^o \leq v^l$ whenever:

$$q^f\sigma\lambda + \delta q^f(1-\lambda\sigma) + \frac{q^f(1-\lambda\sigma)(1-\delta)}{1-\lambda q^f\sigma}(1-\sigma)\lambda = \pi\left[q^f\sigma\lambda + (1-\sigma)\lambda\right]$$

Define $\Psi(q^f, \sigma)$:

$$\Psi(\sigma, q^f) = \pi(\sigma q^f + (1-\sigma)) - q^f\sigma - \delta q^f(1/\lambda - \sigma) - \frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\lambda\sigma}(1-\sigma)$$

That is, $q^f$ is $q^f$ is induced by $\sigma$ and $p^f \in (v^l, v^h)$, $p^o \leq v^l$ when $\Psi(q^f, \sigma) = 0$.

**Lemma 7.** *There exists a decreasing function* $\mathbf{q}^f : [0,1] \to [0,\pi]$*, such that the flagship quality composition* $q^f$ *is induced by* $\sigma$*,* $p^f \in (v^l, v^h)$*,* $p^o \leq v^l$ *if and only if* $q^f = \mathbf{q}^f(\sigma)$*.*

*Proof.* First, let me show that for every $\sigma$, the steady-state flagship average quality is unique and well-defined. To that end, it is sufficient to show that $\Psi(\cdot, \sigma)$ is decreasing in $q^f$ for every $\sigma$ and $\Psi(\cdot, \sigma)$ changes its sign at some interior $q^f$.

Note that we have $\Psi(0, \sigma) = \pi(1-\sigma) \geq 0$, where the inequality is strict if and only if $\sigma < 1$. On the other hand, $\Psi(\pi, \sigma)$ is:

$$\begin{aligned}
\Psi(\pi, \sigma) &= \pi(1-\sigma) - \sigma(1-\pi)\pi - \delta\pi(1/\lambda - \sigma) - (1-\delta)(1-\sigma)\frac{\pi(1-\lambda\sigma)}{1-\pi\sigma\lambda} \\
&= (1-\delta)\pi(1-\sigma) - \delta\pi(1-\lambda)/\lambda - \sigma(1-\pi)\pi - (1-\delta)(1-\sigma)\frac{\pi(1-\lambda\sigma)}{1-\pi\sigma\lambda} \\
&= (1-\delta)\frac{\pi(1-\pi)\lambda\sigma(1-\sigma)}{1-\pi\sigma\lambda} - \delta\pi(1-\lambda)/\lambda - \sigma(1-\pi)\pi \\
&= -\pi(1-\pi)\sigma\frac{\lambda\sigma(1-\pi) + (1-\lambda)}{1-\pi\sigma\lambda} - \delta\frac{\pi(1-\pi)\lambda\sigma(1-\sigma)}{1-\pi\sigma\lambda} - \delta\pi(1-\lambda)/\lambda \leq 0
\end{aligned}$$

Hence, for every $\sigma < 1$ there exist $q^f \in [0,1]$, such that $\Psi(q^f, \sigma) = 0$ (by the Intermediate Value Theorem due to continuity of $\Psi$ in $q^f$).

Now, let me verify that $\Psi(q^f, \sigma)$ is decreasing in $q^f$:

$$\frac{\partial\Psi(q^f, \sigma)}{\partial q^f} = -\sigma(1-\pi) - \delta(1/\lambda - \sigma) - (1-\delta)\frac{(1-\sigma)(1-\lambda\sigma)}{(1-q^f\sigma\lambda)^2} < 0$$

Hence, an intersection with 0 is unique for every $\sigma$. I can denote such an intersection as $\mathbf{q}^f(\sigma)$. Given the two boundaries and the fact $\Psi(q^f, \sigma)$ is decreasing in $q^f$, $\mathbf{q}^f(\sigma) \in [0, \pi]$ for every $\sigma$.

As $\mathbf{q}^f(\sigma)$ is well defined, we can now analyze how the flagship quality composition changes with the flagship customer share. By an Implicit Function Theorem, we have:

$$\frac{\partial \mathbf{q}^f(\sigma)}{\partial \sigma} = -\frac{\partial \Psi(\mathbf{q}^f(\sigma), \sigma)/\partial \sigma}{\partial \Psi(\mathbf{q}^f(\sigma), \sigma)/\partial q^f}$$

Given that $\partial \Psi(\mathbf{q}^f(\sigma), \sigma)/\partial q^f < 0$, the sign of $\partial \mathbf{q}^f(\sigma)/\partial \sigma$ is determined by $\partial \Psi(\mathbf{q}^f(\sigma), \sigma)/\partial \sigma$. As $\mathbf{q}^f(\sigma) \leq \pi$, to establish $\frac{\partial \mathbf{q}^f(\sigma)}{\partial \sigma} < 0$, it suffices to show that $\Psi(q^f, \sigma)$ is decreasing in $\sigma$ for $q^f \leq \pi$.

$$\begin{aligned}
\frac{\partial \Psi(q^f, \sigma)}{\partial \sigma} &= -\pi - q^f(1-\pi-\delta) + (1+\lambda-2\lambda\sigma)(1-\delta)\frac{q^f}{1-\sigma q^f \lambda} \\
&- (1-\delta)\lambda(1-\sigma)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f \lambda}\right)^2 \\
&= -\pi - q^f(1-\pi-\delta) + 2(1-\delta)\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f \lambda} - (1-\delta)\left(\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f \lambda}\right)^2 \\
&- (1-\delta)(1-\lambda)\frac{q^f}{1-\sigma q^f \lambda} + (1-\delta)(1-\lambda)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f \lambda}\right)^2 \\
&\leq^{(1)} -\pi - q^f(1-\pi-\delta) + 2(1-\delta)\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f \lambda} - (1-\delta)\left(\frac{(1-\lambda\sigma)q^f}{1-\sigma q^f \lambda}\right)^2 \\
&\leq^{(2)} -\pi - q^f(1-\pi-\delta) + 2(1-\delta)q^f - (1-\delta)\left(q^f\right)^2 
\end{aligned} \tag{2}$$

where (1) holds because

$$\begin{aligned}
&- (1-\delta)(1-\lambda)\frac{q^f}{1-\sigma q^f \lambda} + (1-\delta)(1-\lambda)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f \lambda}\right)^2 \\
&= (1-\delta)(1-\lambda)q^f\left(\frac{1}{1-\sigma q^f \lambda}\right)^2\left[-(1-q^f\sigma\lambda) + q^f(1-\lambda\sigma)\right] \leq 0
\end{aligned}$$

and (2) holds because $\frac{(1-\lambda\sigma)q^f}{1-\sigma\lambda q^f} \leq q^f$, and $2x - x^2$ increasing in $x$ for $x \leq 1$.

Now, I show that the expression (2) that bounds $\frac{\partial \Psi}{\partial \sigma}$ is increasing in $q^f$. Differentiating it with respect to $q^f$, we get:

$$\frac{\partial\left(-\pi - q^f(1-\pi-\delta) + 2(1-\delta)q^f - (1-\delta)\left(q^f\right)^2\right)}{\partial q^f} = 1 - \delta + \pi - 2(1-\delta)q^f$$

$$\geq 1 - \delta + \pi - 2(1-\delta)\pi$$
$$= (1-\delta)(1-\pi) + \pi\delta > 0$$

Hence, we can bound $\frac{\partial \Psi(q^f,\sigma)}{\partial \sigma}$ by letting $q^f = \pi$ in the boundary of (2):

$$\frac{\partial \Psi(q^f,\sigma)}{\partial \sigma} \leq -\pi - \pi(1-\pi-\delta) + 2(1-\delta)\pi - (1-\delta)\pi^2$$
$$\leq -\pi - \pi(1-\pi) + 2\pi - \pi^2 = 0$$

where the inequality of the second line is due to the fact the expression on the first line is decreasing in $\delta$. □

Next, consider the condition for $q^o$ to be induced by $\sigma, p^f \in (v^l, v^h)$, $p^o \leq v^l$. At the outlet, the outflow of high-quality items includes the purchases $q^o(1-\sigma)\lambda$ and depreciation of the remaining high-quality stock $\delta(1-(1-\sigma)\lambda)q^o$. The inflow is given the after-sales-after-depreciation $\frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\sigma\lambda}$ average quality of the flagship times the total outlet sales $(1-\sigma)$. Equating the outflow of the high-quality items to their inflow at the outlet, we obtain:

$$q^o(1-\sigma)\lambda + \delta(1-(1-\sigma)\lambda)q^o = \frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\sigma\lambda}(1-\sigma)\lambda$$
$$q^o = \frac{1-\sigma}{1-\sigma+\delta(1/\lambda-(1-\sigma))}\frac{q^f(1-\lambda\sigma)(1-\delta)}{1-q^f\sigma\lambda} \tag{3}$$

As the flagship's quality composition is unique for every $\sigma$, so is the outlet's average quality (whenever either the outlet has a positive consumer share $\sigma < 1$ or the depreciation rate $\delta$ is positive). As before, at the steady state, consumers are more pessimistic about the outlet's quality composition relative to the flagship.

**Lemma 8.** *At the induced steady state, the relative quality composition between the flagship and the outlet $q^f/q^o$ increases with the flagship's customer share $\sigma$ for any $\sigma < 1$.*

*Proof.* By Equation (3) and Lemma 1, the ratio between the flagship's and the outlet's quality composition for the given flagship customer share $\sigma < 1$ is:

$$\frac{\mathbf{q}^f(\sigma)}{\mathbf{q}^o(\sigma)} = \frac{1-\mathbf{q}^f(\sigma)\sigma\lambda}{1-\sigma\lambda}\frac{(1-\sigma)(1-\delta)+\delta/\lambda}{1-\sigma}\frac{1}{1-\delta}$$

Differentiating the above with respect to $\sigma$, we get:

$$\frac{\partial \mathbf{q}^f(\sigma)/\mathbf{q}^o(\sigma)}{\partial \sigma} = -\frac{\partial \mathbf{q}^f(\sigma)}{\partial \sigma}\frac{\lambda\sigma}{1-\lambda\sigma}\frac{(1-\sigma)(1-\delta)+\delta/\lambda}{1-\sigma}\frac{1}{1-\delta}$$

$$+\frac{\lambda(1-\mathbf{q}^f(\sigma))}{(1-\sigma\lambda)^2}\frac{(1-\sigma)(1-\delta)+\delta/\lambda}{1-\sigma}\frac{1}{1-\delta}+\frac{1-\mathbf{q}^f(\sigma)\sigma\lambda}{1-\sigma\lambda}\frac{1}{(1-\sigma)^2}\frac{\delta/\lambda}{1-\delta}>0$$

where the inequality follows from $\frac{\partial \mathbf{q}^f(\sigma)}{\partial\sigma}<0$ due to Lemma 1. □

As the indifference condition for the consumers remains the same in the extended model for any prices $p^f \in (v^l, v^h)$, $p^o = v^l$:

$$\frac{\mathbf{q}^f(\sigma)}{\mathbf{q}^o(\sigma)}=\frac{v^h-v^l}{v^h-p^f}$$

The two lemmas above imply the same equilibrium relationship as Theorem 1: The higher the flagship price, the higher the quality composition ratio required to sustain the interior equilibrium. Consequently, more consumers must shop at the flagship by Lemma 8 in the interior equilibrium with a higher flagship price. By Lemma 1 and Equation (3), a higher flagship customer share then leads to deterioration of quality at both stores. As a result, the total per-period sales volume compresses due to the same two effects as in the benchmark two-store model in the main text.

# B Other Prices in the Two-Store Model

In this Appendix, I discuss how the predictions would change for different choices of prices for the two stores.

**Quality Composition Evolution**. Let $S_t^{i,\omega}$ denote the total sales volume of product type $\omega$ at store $i$ in period $t$. I omit the product type index to refer to the total sales volume at a store $i$ as $S_t^i$. Then, generalizing the evolution of the quality composition for the two stores, we obtain:

$$\Delta q_t^o = q_{t,a}^f S_t^o - S_t^{o,h} \tag{4}$$
$$\Delta q_t^f = \pi(S_t^f + S_t^o) - S_t^{f,h} - q_{t,a}^f S_t^o \tag{5}$$

As before, all the sales volumes are determined by the consumer behavior. Consider the flagship store, for example. If its current quality composition is $q_t^f$ and it attracts a $\sigma$ share of consumers, the total number of consumers who inspect a high-quality item is $\lambda\sigma q_t^f$. The flagship store sells all these items if the price is no higher than the consumer value of a high-quality item:

$$S_t^{f,h} = q_t^f\sigma\lambda\mathbb{1}\{p^f \le v^h\}$$

Similarly, the flagship sells to all consumers who find a low-quality product, in the mass of $(1-q_t^f)\sigma\lambda$, if the price is below the low-quality product's value $v^l$:

$$S_t^{f,l} = (1-q_t^f)\sigma\lambda\mathbb{1}\{p^f \leq v^l\}$$

Analogously, the outlet sells a product of type $\omega$ to all consumers who draw such a product if its value exceeds the outlet price:

$$S_t^{o,h} = q_t^o(1-\sigma)\lambda\mathbb{1}\{p^o \leq v^h\}$$
$$S_t^{o,l} = (1-q_t^o)(1-\sigma)\lambda\mathbb{1}\{p^o \leq v^l\}$$

**Consumer Payoff.** Consumers select their shopping strategy to maximize their expected payoff:

$$\begin{aligned} V^B(p^f, p^o, \sigma, q^f, q^o) = &\sigma[q^f(v^h - p^f)_+ + (1-q^f)(v^l - p^o)_+] \\ &+ (1-\sigma)[q^o(v^h - p^o)_+ + (1-q^o)(v^l - p^o)_+] \end{aligned}$$

The shopping strategy $\sigma$ and steady-state quality composition $(q^f, q^o)$ form *a steady-state equilibrium in the consumers' game* given the prices $p^f, p^o$, if the quality composition $(q^f, q^o)$ is induced by $\sigma, p^f, p^o$ and the shopping strategy $\sigma$ is consumer-optimal given prices and the quality composition at each store.

Let $\mathcal{E}_{p^f,p^o}$ denote all possible equilibria in the consumers' game given the prices $p^f, p^o$.
**Seller's Problem**. The seller chooses the prices and any steady-state equilibrium $(\sigma, q^f, q^a) \in \mathcal{E}_{p^f,p^o}$ to maximize per-customer steady-state profit flow in both stores:

$$V^S(p^f, p^o, \sigma, q^f, q^o) = p^f S^f + p^o S^o$$

**Induced Steady States under Other Prices**. First, note that if $p^o \in (v^l, v^h)$, then in any induced steady-state $q^o(1-\sigma) = 0$. Indeed, in this case, the outlet sells high-quality items only. Then, from Equation (4), $q^o$ is induced by $\sigma$ and $p^f, p^o$ if: $q^o(1-\sigma) = q_a^f(1-\sigma)$, which can only be satisfied if $q^o(1-\sigma) = 0$. Hence, the outlet must offer low prices to have any steady-state sales. It is useless to the seller otherwise.

If $p^f \leq v^l$, then in any induced steady-state $q^f[\sigma + S^o] = \pi[\sigma + S^o]$. In this case, the flagship's total sales is $\sigma$, and a share $q^f$ of these is of high quality. Then, the after-sales proportion of high-quality products is the same as before consumer arrival. From

Equation (4), we must have:

$$q^f \sigma + q^f S^o = \pi[\sigma + S^o]$$

Hence, if $p^f \leq v^l$, either the seller makes zero sales or the flagship store has a share $\pi$ of high-quality products. The seller can do no better than running a single outlet store at a price $v^l$ in the baseline case for the store prices.

Hence, we may restrict attention to prices $p^f \in (v^l, v^h)$ and $p^o \leq v^l$. Now, let me verify that $p^o = v^l$ is without loss for the seller. The seller's payoff is given by:

$$\sigma q^f p^f + (1-\sigma) q^o p^o$$

If $\sigma = 0$, then the seller earns $\pi p^o \leq \pi v^l$. If $\sigma = 1$, the seller makes no sales by Lemma 2, and any such outcome is not optimal. It remains to verify that $p^o = v^l$ is without loss for the potential outcomes with interior $\sigma$. In this case, the prices must satisfy the consumers' indifference condition:

$$q^f (v^h - p^f) = q^o (v^h - v^l) + (v^l - p^o)$$

If $p^o < v^l$, then the seller can increase both prices to satisfy the indifference condition for the same $q^f, q^o$. Then, $\sigma, q^f, q^o$ is an equilibrium given the new prices, but the seller improves upon her profit.

# C Seller's Problem: Two-Store Model

*Proof of Proposition 2.* The seller's payoff in an interior equilibrium for some flagship price $p^f \in (v^l, v^h)$ and the outlet price $p^o = v^l$ is given by:

$$V^S(p^f, \boldsymbol{\sigma}(p^f), \mathbf{q}^f(\boldsymbol{\sigma}(p^f))) = \boldsymbol{\sigma}(p^f)\mathbf{q}^f(\boldsymbol{\sigma}(p^f))p^f + (1 - \boldsymbol{\sigma}(p^f))v^l$$

The interior equilibrium condition requires $\mathbf{q}^f(\boldsymbol{\sigma}(p^f))p^f = \mathbf{q}^f(\boldsymbol{\sigma}(p^f))v^h - \mathbf{q}^o(\boldsymbol{\sigma}(p^f))(v^h - v^l)$, so that we may rewrite the seller's problem as follows:

$$V^S(p^f, \boldsymbol{\sigma}(p^f), \mathbf{q}^f(\boldsymbol{\sigma}(p^f))) = \boldsymbol{\sigma}(p^f)\mathbf{q}^f(\boldsymbol{\sigma}(p^f))v^h + (1 - \boldsymbol{\sigma}(p^f))v^l - \boldsymbol{\sigma}(p^f)\mathbf{q}^o(\boldsymbol{\sigma}(p^f))(v^h - v^l)$$

If the seller instead only operates an outlet, she receives a payoff of $v^l$. Hence, the seller prefers to operate both stores with a flagship price $p^f$ whenever:

$$\begin{aligned} V^S(p^f, \boldsymbol{\sigma}(p^f), \mathbf{q}^f(\boldsymbol{\sigma}(p^f))) &= \boldsymbol{\sigma}(p^f)\mathbf{q}^f(\boldsymbol{\sigma}(p^f))v^h + (1-\boldsymbol{\sigma}(p^f))v^l - \boldsymbol{\sigma}(p^f)\mathbf{q}^o(\boldsymbol{\sigma}(p^f))(v^h - v^l) \\ &> v^l = V^S(p^f, 0, \pi) \end{aligned}$$

which holds true if and only if:

$$\mathbf{q}^f(\boldsymbol{\sigma}(p^f))v^h - \mathbf{q}^o(\boldsymbol{\sigma}(p^f))v^h - (1 - \mathbf{q}^o(\boldsymbol{\sigma}(p^f)))v^l > 0 \tag{6}$$

(i) Take some price $p^f \in (v^h, v^l)$, and fix the equilibrium in the consumer's game to be $\sigma = \boldsymbol{\sigma}(p^f|v^h), q^f = \mathbf{q}^f(\boldsymbol{\sigma}(p^f|v^h)), q^o = \mathbf{q}^o(\boldsymbol{\sigma}(p^f|v^h))$. Suppose $v^h$ increases to $\tilde{v}^h > v^h$. Adjust the price to a new level $\tilde{p}^f$, so that to obtain the same interior equilibrium in the consumers' game:

$$\frac{\tilde{v}^h - v^l}{\tilde{v}^h - \tilde{p}^f} = \frac{q^f}{q^o}$$

Clearly, such a price exists with $\tilde{p}^f \in (v^l, \tilde{v}^h)$.

First, note that provided $\tilde{v}^h$ is high enough, Equation (6) (since $q^f > q^o$). In addition, if the seller prefers to operate two stores with a flagship price $p^f$ at the high-quality item's value $v^h$, she must also prefer to operate both stores with a flagship price $\tilde{p}^f$ when the value increases to $\tilde{v}^h$.

In addition, Equation (6) cannot be satisfied for $v^h = v^l$. The result of (i) follows.

(ii) The seller's payoff from operating both stores is bounded above by $\pi v^h + (1-\pi)v^l$ (strictly so whenever $\pi$ is interior). If $\pi \to 0$, this bound equals $v^l$. Hence, if $\pi \approx 0$, the seller prefers to operate the outlet only.

As in the previous step, take some price $p^f \in (v^h, v^l)$, and fix the equilibrium in the consumer's game to be $\sigma = \boldsymbol{\sigma}(p^f|\pi), q^f = \mathbf{q}^f(\boldsymbol{\sigma}(p^f|\pi)|\pi), q^o = \mathbf{q}^o(\boldsymbol{\sigma}(p^f|\pi)|\pi)$. Suppose the proportion of high-quality items increases from $\pi$ to $\tilde{\pi} > \pi$.

I now show that $\boldsymbol{\sigma}(p^f|\pi) > \boldsymbol{\sigma}(p^f|\tilde{\pi})$. As the price remains the same, it must be that in the equilibrium under the new high-quality goods share, the relative flagship's premium is the same. By the same proof as in Lemma 7, $\partial\mathbf{q}^f(\sigma|\pi)/\partial\pi$ is determined by the sign of $\partial\Psi/\partial\pi$, which is positive.

Recall that:

$$\frac{q^f}{q^o} = \frac{1 - q^f\sigma}{1 - \sigma}$$

which is decreasing in $q^f$ and increasing in $\sigma$. Since $\mathbf{q}^f(\cdot|\cdot)$ is decreasing in $\sigma$ but increasing in $\pi$, to preserve the same relative quality premium at the flagship, as $\pi$ increases to $\tilde{\pi}$, the flagship customer share must rise: $\boldsymbol{\sigma}(p^f|\pi) > \boldsymbol{\sigma}(p^f|\tilde{\pi})$. The net effect on $q^f$ of the two changes (in $\sigma$ and $\pi$) must be positive: $\mathbf{q}^f(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi}) > \mathbf{q}^f(\boldsymbol{\sigma}(p^f|\pi)|\pi)$, or else the relative quality premium would strictly decrease.

As the ratio of the two qualities is the same, then the outlet's quality also increases: $\mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi}) > \mathbf{q}^o(\boldsymbol{\sigma}(p^f|\pi)|\pi)$. Hence, if Equation (6) is satisfied at $p^f$ given $\pi$, then it is also satisfied at $p^f$ given $\tilde{\pi}$:

$$\begin{aligned}
&\tilde{\mathbf{q}}^f(\boldsymbol{\sigma}(p^f))v^h - \tilde{\mathbf{q}}^o(\boldsymbol{\sigma}(p^f))v^h - (1 - \tilde{\mathbf{q}}^o(\boldsymbol{\sigma}(p^f)))v^l = \\
&\tilde{\mathbf{q}}^o(\boldsymbol{\sigma}(p^f))\left(q^f/q^o v^h - v^l\right) - (1 - \tilde{\mathbf{q}}^o(\boldsymbol{\sigma}(p^f)))v^l \\
&\geq \mathbf{q}^o(\boldsymbol{\sigma}(p^f))\left(q^f/q^o v^h - v^l\right) - (1 - \mathbf{q}^o(\boldsymbol{\sigma}(p^f)))v^l > 0
\end{aligned}$$

It only remains to show that the seller prefers to offer two stores when $\tilde{\pi} \to 1$.

By definition of $\mathbf{q}^f$, $\mathbf{q}^o$:

$$\pi[\mathbf{q}^f(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi})\boldsymbol{\sigma}(p^f|\tilde{\pi}) + 1 - \boldsymbol{\sigma}(p^f|\tilde{\pi})] - \mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi})(1 - \boldsymbol{\sigma}(p^f|\tilde{\pi})) - \boldsymbol{\sigma}(p^f|\tilde{\pi})\mathbf{q}^f(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi}) = 0$$

By our construction, we must have:

$$\frac{\tilde{\pi}(1 - \boldsymbol{\sigma}(p^f|\tilde{\pi})))}{\mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi})} - (1 - \boldsymbol{\sigma}(p^f|\tilde{\pi}))) - \frac{q^f}{q^o}\boldsymbol{\sigma}(p^f|\tilde{\pi}))(1 - \tilde{\pi}) = 0$$

As $\tilde{\pi} \to 1$, the above implies:

$$\lim_{\tilde{\pi}\to 1}(1 - \boldsymbol{\sigma}(p^f|\tilde{\pi})))\frac{\tilde{\pi} - \mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})}{\mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi})} = 0$$

implying that either $\lim_{\tilde{\pi}\to 1}\mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi} = 1$ or $\lim_{\tilde{\pi}\to 1}\boldsymbol{\sigma}(p^f|\tilde{\pi})) = 1$. In the first case, the left-hand side of Equation (6) converges to $v^h(q^f/q^o - 1) > 0$ and we are done.

I now consider the second case. Recall that we must have:

$$\frac{q^f}{q^o} = \frac{1 - \mathbf{q}^f(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi})\boldsymbol{\sigma}(p^f|\tilde{\pi})}{1 - \boldsymbol{\sigma}(p^f|\tilde{\pi})}$$

Then, if $\lim_{\tilde{\pi}\to 1}\boldsymbol{\sigma}(p^f|\tilde{\pi})) = 1$, then $\lim_{\tilde{\pi}\to 1}\mathbf{q}^f(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi}) = 1$, which implies

$$\lim_{\tilde{\pi}\to 1}\mathbf{q}^o(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi} = \frac{q^o}{q^f}\lim_{\tilde{\pi}\to 1}\mathbf{q}^f(\boldsymbol{\sigma}(p^f|\tilde{\pi})|\tilde{\pi}) = \frac{q^o}{q^f}$$

In this case, Equation (6) converges to:

$$\frac{q^o}{q^f}\left(\frac{q^f}{q^o}-1\right)v^h-\left(1-\frac{q^o}{q^f}\right)v^l=\left(1-\frac{q^o}{q^f}\right)(v^h-v^l)>0$$

$\square$

*Proof of Proposition 3.* **Step 1**. For any flagship price $p^f$, the flagship-seller's payoff increases in $\alpha$. Hence, if the two stores are operational at $\alpha$, they must also be operational with smaller frictions of vertical integrations $\tilde{\alpha}>\alpha$.
**Step 2**. The flagship-seller's payoff has increasing differences in $(\alpha,-p^f)$. Indeed, let:

$$V^S(p^f,\boldsymbol{\sigma}(p^f),\mathbf{q}^f(\boldsymbol{\sigma}(p^f)),\mathbf{q}^o(\boldsymbol{\sigma}(p^f))|\alpha)=\boldsymbol{\sigma}(p^f)\mathbf{q}^f(\boldsymbol{\sigma}(p^f))p^f+\alpha v^l(1-\boldsymbol{\sigma}(p^f))$$

Then, taking the mixed derivative in $(p^f,\alpha)$, we get:

$$\frac{\partial^2 V^S(p^f,\boldsymbol{\sigma}(p^f),\mathbf{q}^f(\boldsymbol{\sigma}(p^f)),\mathbf{q}^o(\boldsymbol{\sigma}(p^f))|\alpha)}{\partial\alpha\partial p^f}=-v^l\frac{\partial\boldsymbol{\sigma}(p^f)}{\partial p^f}<0$$

where the inequality follows from Theorem 1(i), as $\frac{\partial\boldsymbol{\sigma}(p^f)}{\partial p^f}>0$. Then, by Milgrom and Shannon (1994), the optimal flagship price decreases in $\alpha$. $\square$

# D Omitted Proofs for Section 3

**Lemma 9.** *For every admissible price schedule,* $\mathbf{p}:X\to\mathbb{R}$*, there exists an equilibrium in the consumers' game* $\langle\sigma,\mathbf{q}\rangle$ *given* $\mathbf{p}$*, such that* $\sigma(x)>0$ *only if* $x\in \underset{y\in X}{argmin}\,\mathbf{p}(y)$.

*Proof of Lemma 9.* Recall that admissibility requires that the price schedule $\mathbf{p}$, reaches its minimum and remains constant on some interval of size $\varepsilon>0$. Suppose that $(x_1,x_2]$ is such an interval.

$$\sigma(x)=\begin{cases}\frac{1}{\varepsilon}, & \text{if } x\in(x_1,x_2]\\ 0, & \text{else}\end{cases}$$

First, note that the suggested strategy is admissible (corlol). For the steady-state quality composition, we get two cases:
*Case 1.* Suppose that $\underline{p}>v^l$. Then, we can construct a buyer's equilibrium with zero sales. For instance, $\mathbf{q}(x)\equiv 0$ for all $x\in X$. In this equilibrium, $S(x)=0$, as neither location offers goods of high quality, and the price is prohibitively high to purchase any of the low-quality

items. The steady-state condition is satisfied since, with zero sales, as there is no movement in stock across the locations, we can sustain any steady-state quality composition.

*Case 2.* Suppose that $\underline{p} \leq v^l$. Then, $\mathbf{q}(x) \equiv \pi$ satisfies the steady-state condition. □

# Appendices for the Continuous Model

## E General Model

In this section, I formally describe the most general version of the continuous model, which allows buyers to have a more general shopping strategy, allows for direct disposal, and considers multiple quality tiers.

**Quality Tiers.** Suppose there are $n$ quality tiers for the product. I assume that $v_i$ is a consumer's product value $i$ net of the seller's production cost $\kappa$. For notational simplicity, I assume that the consumer values (net of production costs) are ordered in decreasing order: $v^1 > v^2 > \ldots v^n$. In addition, it will be useful to define a fictitious product $n+1$ with a value $v^{n+1} = -\infty$.

**Replenishment with Direct Disposal.** As in the main body of the paper, the seller manages a continuum of shopping locations $X = (0, 1)$ and a warehouse at location 1, which cannot be visited by the consumers and only stores products before they are disposed of. The products are sent away from location 1 at a constant rate $\gamma \geq 0$.

**Prices and Quality Composition**. The quality composition is now described by $\mathbf{q} : \{1, \ldots, n\} \times X \to [0, 1]^n$, with $\mathbf{q}(i|x)$ denoting the proportion of product $i$ in the stock of location $x$. The quality composition at the production plant is given exogenously with some $\{\pi(i)\}_{i=1}^n$, so that for every steady-state quality composition $\mathbf{q}(i|0) = \pi(i)$. I assume the seller produces each quality $i$ with a positive probability $\pi(i) \in (0, 1)$. Let $\mathbf{p} : X \to \mathbb{R}$ summarize the price schedule for locations in $X$, with $\mathbf{p}(x)$ being the price the seller receives (conditional on purchase) net of her replacement cost $\kappa > 0$. I assume that $\mathbf{p}$ and every $\mathbf{q}(i|\cdot)$ are Lebesgue-measurable.

**Consumers.** Consumers who shop at location $x$, draw a single product at random according to distribution $\{\mathbf{q}(i|x)\}_{i=1}^n$. As before, I assume that when indifferent, the consumer makes a purchasing decision about a found product in favor of the seller. The shopping strategy by the consumers is summarized by a cdf $D : [0, 1] \to [0, 1]$—potentially discontinuous, where $D(x)$ denotes the mass of consumers shopping at the locations weakly below $x$. Note that in the main body of the paper, I consider a special case where $D$ admits a density: there exists some integrable $\sigma : X \to \mathbb{R}_+$ such that:

$$D(x) = \int_{y \leq x} \sigma(y)dy$$

Given that $D$ is a cdf over $[0,1]$, I implicitly assume that $D(0) = 0, D(1) = 1$, and $D$ is continuous on the right with a limit on the left (corlol) on $[0,1]$. In addition, since I assume that location 1 is not available for consumers, I require that $D$ is continuous at 1. Define $\delta : X \to [0,1]$ to be the size of an atom at location $x$: $\delta(x) = D(x) - D(x-)$.

**Induced Steady State.** A market outcome is summarized by a tuple $m = \langle \mathbf{p}, D, \mathbf{q}, \gamma \rangle$. For every market outcome, we can define the purchasing probability per customer $\rho_m : X \to [0,1]$ for every location as $\rho_m(x) = \sum_{i=1}^{n} \mathbb{1}\{\mathbf{p}(x) \leq v^i\}\mathbf{q}(i|x)$ and the total steady-state downstream sales in a market outcome $m = \langle \mathbf{p}, D, \mathbf{q}, \gamma \rangle$ $S_m : [0,1] \to \mathbb{R}_+$ now become:

$$S_m(x) = \int_{y > x} \rho_m(x)dD(x) + \gamma \mathbb{1}\{x < 1\}$$

I say that $(D, \mathbf{p}, \gamma)$ *induce* a steady-state quality composition $\mathbf{q}$ *on an interval* $[y_1, y_2]$ if for every $i$ and every $(x_1, x_2] \subset [y_1, y_2]$:

$$\int_{y \in (x_1, x_2], p(y) \leq v^i} \mathbf{q}(i|y)dD(y) = S_m(x_1)\mathbf{q}(i|x_1) - S_m(x_2)\mathbf{q}(i|x_2)$$

**Payoffs**. The payoffs of a buyer and the seller are given respectively by:

$$V^B(\mathbf{p}, D, \mathbf{q}, \gamma) = \int \sum_{i=1}^{n} \mathbf{q}(i|x)(v^i - \mathbf{p}(x))_+ dD(x)$$
$$V^S(\mathbf{p}, D, \mathbf{q}, \gamma) = \int \mathbf{p}(x)\rho_m(x)dD(x) - \kappa\gamma$$

I now adapt the definitions of equilibrium in the consumer's game and the set of market outcomes to this generalized set-up.

**Definition 2*.** Say that a shopping strategy $D : [0,1] \to [0,1]$ and a steady-state quality composition across the locations $\mathbf{q} : \{1, \ldots, n\} \times [0,1] \to [0,1]^n$ is *an equilibrium in the consumers' game* given a price schedule $\mathbf{p} : X \to \mathbb{R}_+$ and disposal rate $\gamma \geq 0$

(i) $V^B(\mathbf{p}, D, \mathbf{q}, \gamma) = \max_{x \in X} \sum_{i=1}^{n} \mathbf{q}(i|x)(v^i - \mathbf{p}(x))_+ dD(x)$

(ii) $\mathbf{q}$ is induced by $(D, \mathbf{p}, \gamma)$

Let $\mathcal{E}_{\mathbf{p},\gamma}$ denote the set of all possible equilibria in the consumers' game given prices $\mathbf{p}$ and disposal rate $\gamma$.

In the remainder of the Appendix, I slightly abuse the notation and write $\langle \mathbf{p}, \sigma, \mathbf{q}, \gamma \rangle$ to denote a market outcome where the shopping strategy of a consumer admits a density, or write $\langle \mathbf{p}, D, \mathbf{q} \rangle$ to denote a market outcome where $\gamma = 0$.

# F Proofs for the General Model

In this appendix, I prove the results for the General Model outlined in Appendix E.

**Lemma 10.** *Consider a market outcome $m = \langle \mathbf{p}, D, \mathbf{q} \rangle$. $\mathbf{q}$ is induced by $D$, $\mathbf{p}, \gamma$ only if it satisfies the following:*

*i) if $S_m(x) > 0$, then for every $i$, $\mathbf{q}(i|\cdot)$ is right-continuous at $x$, and is continuous at $x$ if $D$ is continuous at $x$.*

*ii) if $\mathbf{p}(x) \leq v^n$ or $\mathbf{p}(x) > v^1$, then $\mathbf{q}(i|\cdot)$ is left-continuous at $x$ whenever $S_m(x-) > 0$*

*iii) if $\mathbf{p}(x) > v^i$, then $S_m(\cdot)\mathbf{q}(i|\cdot)$ is continuous at $x$*

*iv) if $\mathbf{p}(x) < v^l$ and $D$ is discontinuous at $x$, then $\mathbf{q}(i|x) < \mathbf{q}(i|x-)$ whenever $\mathbf{q}(i|x-) > 0$ and $\rho_m(x) < 1$*

*Proof.* (i) $\mathbf{q}$ is induced by $(D, \mathbf{p}, \gamma)$ on $(x, x+\Delta]$ if:

$$-\int_{y \in (x, x+\Delta], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y) + S_m(x)\mathbf{q}(i|x) - S_m(x+\Delta)\mathbf{q}(i|x+\Delta) = 0$$

and we can take $\Delta$ to be arbitrarily small. Note that $\int_{y \in (x, x+\Delta], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y)$ converges to 0 by the Squeeze Theorem:

$$0 = \lim_{\Delta \to 0} \int_{y \in (x, x+\Delta]} 1 dD(y) \geq \int_{y \in (x, x+\Delta], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y) \geq 0$$

where the equality is due to $D$ being right-continuous at every $x \in X$. Hence, $\mathbf{q}$ is induced by $D$, $\mathbf{p}, \gamma$ on $(x, x+\Delta]$ for any small $\Delta$:

$$S_m(x)\mathbf{q}(i|x) = \lim_{\Delta \to 0} S_m(x+\Delta)\mathbf{q}(i|x+\Delta)$$

As $S_m$ is right-continuous at every $x$, so is $\mathbf{q}(i|x)$ for any $x$, such that $S_m(x) > 0$. Analogously, we can show that if $D$ is left-continuous at $x$, then $\mathbf{q}(i|x)$ must be left-continuous if $S_m(x) > 0$.

(ii) If $\mathbf{p}(x) \leq v^n$, $\mathbf{q}$ being induced by $(D, \mathbf{p}, \gamma)$ on $(x-\Delta, x]$ for any small $\Delta$ and for every $1 \leq i \leq n$ requires:

$$\begin{aligned}S_m(x-\Delta)\mathbf{q}(i|x-\Delta) - S_m(x)\mathbf{q}(i|x) &= \int_{y\in(x-\Delta,x],\mathbf{p}(y)\leq v^i} \mathbf{q}(i|y)dD(y)\\ &= \mathbf{q}(i|x)\left(S_m(x-\Delta) - S_m(x)\right)\\ &+ \int_{y\in(x-\Delta,x),\mathbf{p}(y)\leq v^i} \left(\mathbf{q}(i|y) - \mathbf{q}(i|x)\right) dD(y)\\ &- \int_{y\in(x-\Delta,x),\mathbf{p}(y)>v^n} \mathbf{q}(i|x)(1-\rho_m(y))dD(y)\end{aligned}$$

Which we can rewrite as:

$$\begin{aligned}S_m(x-\Delta)(\mathbf{q}(i|x-\Delta) - \mathbf{q}(i|x)) &= \int_{y\in(x-\Delta,x),\mathbf{p}(y)\leq v^i} \left(\mathbf{q}(i|y) - \mathbf{q}(i|x)\right) dD(y)\\ &- \int_{y\in(x-\Delta,x),\mathbf{p}(y)>v^n} \mathbf{q}(i|x)(1-\rho_m(y))dD(y)\end{aligned}$$

Note that the right-hand side is converging to 0 as $\Delta \to 0$, as $|\mathbf{q}(i|y) - \mathbf{q}(i|x)|$ and $\mathbf{q}(i|x)(1-\rho_m(y))$ are at most 1. Hence, we obtain that:

$$S_m(x-\Delta)(\mathbf{q}(i|x-\Delta) - \mathbf{q}(i|x)) \underset{\Delta\to 0}{\rightarrow} 0$$

That is, unless $S_m(x-) = 0$, $\mathbf{q}(i|\cdot)$ is left-continuous at $x$.

The proof for the case $\mathbf{p}(x) > v^1$ is analogous.

(iii) Suppose that $\mathbf{p}(x) > v^i$. Then, $S_m(\cdot)\mathbf{q}(i|\cdot)$ is continuous at $x$. Indeed, in this case, we obtain:

$$S_m(x-\Delta)\mathbf{q}(i|x-\Delta) - S_m(x)\mathbf{q}(i|x) = \int_{\mathbf{p}(x)\leq v^i, y\in(x-\Delta,x)} \mathbf{q}(i|y)dD(y) \underset{\Delta\to 0}{\rightarrow} 0$$

(iv) Suppose that $\mathbf{p}(x) < v^i$, then $\mathbf{q}$ is induced over $(x-\Delta, x]$ whenever:

$$\begin{aligned}S_m(x-\Delta)\mathbf{q}(i|x-\Delta) - S_m(x)\mathbf{q}(i|x) &= \int_{\mathbf{p}(x)\leq v^i, y\in(x-\Delta,x)} \mathbf{q}(i|y)dD(y)dy\\ &= \delta(x)\mathbf{q}(i|x) + \int_{\mathbf{p}(x)\leq v^i, y\in(x-\Delta,x)} \mathbf{q}(i|y)dD(y)dy\end{aligned}$$

Taking the limit of both sides as $\Delta \to 0$, we obtain:

$$S_m(x-)\mathbf{q}(i|x-) - S_m(x)\mathbf{q}(i|x) = \delta(x)\mathbf{q}(i|x)$$

As $\mathbf{p}(x) \leq v^i$, then $S_m(x-) = \delta(x)\rho_m(x) + S_m(x)$ and the above implies:

$$\delta(x)\mathbf{q}(i|x) + S_m(x)\mathbf{q}(i|x) \leq \mathbf{q}(i|x-)\left[\mathbf{q}(i|x)\delta(x) + S_m(x)\right]$$
$$\mathbf{q}(i|x) = \mathbf{q}(i|x-)\frac{\delta(x)\rho_m(x) + S_m(x)}{\delta(x) + S_m(x)}$$

which is strictly lower than $\mathbf{q}(i|x-)$ whenever $\mathbf{q}(i|x-) > 0$ and $\rho_m(x) < 1$ □

**Lemma 11.** *Consider a market outcome $m = \langle \mathbf{p}, D, \mathbf{q}\rangle$. Suppose that $\mathbf{p}(x) \in (v^{i+1}, v^i)$ D-a.s. on $[x_1, x_2]$, $S_m(x_2) > 0$, and either $D$ is continuous on $[x_1, x_2]$, or $i = n$. Then, $\mathbf{q}$ is induced by $(D, \mathbf{p}, \gamma)$ over $[x_1, x_2]$ if and only if:*

*i) for every $l > i$, $S_m(x)\mathbf{q}(l|x)$ is constant over $[x_1, x_2]$*

*ii) for every $l \leq i$, $\frac{\mathbf{q}(l|x)}{\sum_{k\leq i}\mathbf{q}(k|x)}$ is constant over $[x_1, x_2]$* [17]

*Proof.* $\mathbf{q}$ is induced by $(D, \mathbf{p}, \gamma)$ over $[x_1, x_2]$ whenever for any $x \in (x_1, x_2]$, any $\Delta > 0$ and $\forall l \in \{1, \ldots, n\}$:

$$\int_{y\in(x-\Delta,x],\mathbf{p}(y)\leq v^l} \mathbf{q}(l|y)dD(y) = S_m(x_1)\mathbf{q}(l|x_1) - S_m(x_1)\mathbf{q}(l|x_1) \tag{7}$$

(i) Take any quality type $l > i$. Then, $\int_{y\in(x-\Delta,x],\mathbf{p}(y)\leq v^l} \mathbf{q}(l|y)dD(y) = 0$, and the above holds if and only if:

$$S_m(x-\Delta)\mathbf{q}(l|x-\Delta) = S_m(x)\mathbf{q}(l|x)$$

(ii) First, let me show the only if direction. Fix some product quality $l \leq i$ and assume by way of contradiction that there exists some $\tilde{x}_1$ and $\tilde{x}_2$, such that $\frac{\mathbf{q}(l|\tilde{x}_1)}{\sum_{k\leq i}\mathbf{q}(k|\tilde{x}_1)} > \frac{\mathbf{q}(l|\tilde{x}_2)}{\sum_{k\leq i}\mathbf{q}(k|\tilde{x}_2)}$ (the other case is symmetric). Given the continuity of $\mathbf{q}(k|\cdot)$ for every $k \in \{1, \ldots, n\}$ on $[x_1, x_2]$ by Lemma 10, there exists some $\tilde{y}_1$, such that for all $y \in (\tilde{y}_1, \tilde{x}_2]$, $\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\leq i}\mathbf{q}(k|\tilde{y}_1)} > \frac{\mathbf{q}(l|y)}{\sum_{k\leq i}\mathbf{q}(k|y)}$. $\mathbf{q}$ is induced over $(\tilde{y}_1, \tilde{x}_2]$ by $D, \mathbf{p}, \gamma$ only if:

$$\begin{aligned}
0 &= -\int_{y\in(\tilde{y}_1,\tilde{x}_2],\mathbf{p}(y)\leq v^l} \mathbf{q}(l|y)dD(y) + S_m(\tilde{y}_1)\mathbf{q}(l|\tilde{y}_1) - S_m(\tilde{x}_2)\mathbf{q}(l|\tilde{x}_2) = \\
&= -\int_{y\in(\tilde{y}_1,\tilde{x}_2],\mathbf{p}(y)\in(v^i,v^{i+1})} \mathbf{q}(l|y)dD(y) + S_m(\tilde{y}_1)\mathbf{q}(l|\tilde{y}_1) - S_m(\tilde{x}_2)\mathbf{q}(l|\tilde{x}_2) \\
&= -\int_{y\in(\tilde{y}_1,\tilde{x}_2],\mathbf{p}(y)\in(v^i,v^{i+1})} \frac{\mathbf{q}(l|y)}{\sum_{k\leq i}\mathbf{q}(k|y)}\rho_m(y)dD(y) + S_m(\tilde{y}_1)\sum_{k\leq i}\mathbf{q}(k|\tilde{y}_1)\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\leq i}\mathbf{q}(k|\tilde{y}_1)}
\end{aligned}$$

[17] With a convention that $\frac{\mathbf{q}(j|x)}{\sum_{k\leq i}\mathbf{q}(k|x)} = 1$ when $\sum_{k\leq i}\mathbf{q}(k|x) = 0$.

$$
\begin{aligned}
&- S_m(\tilde{x}_2)\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)\frac{\mathbf{q}(l|\tilde{x}_2)}{\mathbf{q}(k|\tilde{x}_2)}\\
&>^{(1)} -\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}\left(S_m(\tilde{y}_1)-S_m(\tilde{x}_2)\right)+S_m(\tilde{y}_1)\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}\\
&- S_m(\tilde{x}_2)\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)\frac{\mathbf{q}(l|\tilde{x}_2)}{\mathbf{q}(k|\tilde{x}_2)}\\
&= -\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)S_m(\tilde{y}_1)+\frac{\mathbf{q}(l|\tilde{x}_2)}{\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)}\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)S_m(\tilde{x}_2)\\
&+ S_m(\tilde{x}_2)\left(\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}-\frac{\mathbf{q}(l|\tilde{x}_2)}{\mathbf{q}(k|\tilde{x}_2)}\right)\\
&=^{(2)} \left(\frac{\mathbf{q}(l|\tilde{x}_2)}{\sum_{k\le i}\mathbf{q}(k|\tilde{x}_2)}-\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}\right)\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)S_m(\tilde{x}_2)+S_m(\tilde{x}_2)\left(\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}-\frac{\mathbf{q}(l|\tilde{x}_2)}{\mathbf{q}(k|\tilde{x}_2)}\right)\\
&= S_m(\tilde{x}_2)\left(1-\sum_{k>i}\mathbf{q}(k|\tilde{x}_2)\right)\left(\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}-\frac{\mathbf{q}(l|\tilde{x}_2)}{\mathbf{q}(k|\tilde{x}_2)}\right)>^{(3)} 0
\end{aligned}
$$

where (1) and (3) use $\frac{\mathbf{q}(l|y)}{\sum_{k\le i}\mathbf{q}(k|y)} < \frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k\le i}\mathbf{q}(k|\tilde{y}_1)}$ for all $\tilde{x}_2 \ge y > \tilde{y}_1$. The equality (2) follows from using part (i), which implies $\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)S_m(\tilde{y}_1) = \sum_{k>i}\mathbf{q}(k|\tilde{x}_2)S_m(\tilde{x}_2)$. We then obtain a contradiction.

To prove the if direction, suppose that $\frac{\mathbf{q}(l|y)}{\sum_{k\le i}\mathbf{q}(k|y)}$ remains constant, then Equation (7) becomes:

$$
\begin{aligned}
0 &= -\int_{y\in(x-\Delta,x],\mathbf{p}(y)\in(v^i,v^{i+1})}\frac{\mathbf{q}(l|y)}{\sum_{k\le i}\mathbf{q}(k|y)}\rho_m(y)dD(y)+S_m(x-\Delta)\mathbf{q}(l|x-\Delta)-S_m(x)\mathbf{q}(l|x)\\
0 &= -\frac{\mathbf{q}(l|x)}{\sum_{k\le i}\mathbf{q}(k|x)}(S_m(x-\Delta)-S_m(x))+S_m(x-\Delta)\sum_{k\le i}\mathbf{q}(k|x-\Delta)\frac{\mathbf{q}(l|x)}{\sum_{k\le i}\mathbf{q}(k|x)}\\
&\quad - S_m(x)\sum_{k\le i}\mathbf{q}(k|x)\frac{\mathbf{q}(l|x)}{\sum_{k\le i}\mathbf{q}(k|x)}\\
0 &= \frac{\mathbf{q}(l|y)}{\sum_{k\le i}\mathbf{q}(k|y)}\left[S_m(x)\sum_{k>i}\mathbf{q}(k|x)-S_m(x-\Delta)\sum_{k>i}\mathbf{q}(k|x-\Delta)\right]
\end{aligned}
$$

where the equality is true given the premise of part (i).

□

**Lemma 3*.** Consider any market outcome with $m = \langle \mathbf{p}, D, \mathbf{q}, \gamma\rangle$ with $\mathbf{p}\in\mathcal{A}$ and $(D,\mathbf{q})\in\mathcal{E}_{\mathbf{p},\gamma}$. Suppose that $\mathbf{p}(x) > v^i$ for all $x < \hat{x}$. If $S_m(\hat{x}-) = 0$, then $S_m(0) = 0$.

*Proof.* Suppose $S_m(\hat{x}-) = 0$ but $S_m(0) > 0$. Then, by Lemma 11 (i), $S_m(x)\mathbf{q}(i|x)$ is constant on $(0, \hat{x})$ and is continuous at 0. Hence, we must have:

$$S_m(0)\pi(i) = S_m(\hat{x}-)\mathbf{q}(i|\hat{x}-)$$

By assumption, $\pi(i) \in (0, 1)$, and we obtain a contradiction. □

**Lemma 12.** *Consider a market outcome* $m = \langle \mathbf{p}, D, \mathbf{q}, \gamma \rangle$ *with* $\mathbf{p} \in \mathcal{A}$ *and* $(D, \mathbf{q}) \in \mathcal{E}_{\mathbf{p},\gamma}$ *that has positive total sales:* $S_m(0) > 0$. *Suppose that* $\mathbf{p}(x) > v^i$ *for all* $x < \hat{x}$, *then* $\sum_{k \geq i} \mathbf{q}(k|\hat{x}-) < 1$.

*Proof.* Suppose the statement is not true. First, suppose that there exists some location $x < \hat{x}$ such that $\sum_{k \geq i} \mathbf{q}(k|x) = 1$. Let

$$\tilde{x} = \inf\{x < \hat{x} : \sum_{k \geq i} \mathbf{q}(k|x) = 1\}$$

Note that it must be that $\sum_{k \geq i} \mathbf{q}(k|x) = 1$ for all $x \in [\tilde{x}, \hat{x})$. $\mathbf{q}$ is induced by $D, \mathbf{p}, \gamma$ only if $\sum_{k \geq i} \mathbf{q}(k|x) S_m(x)$ remains constant over $[0, \hat{x})$ (by same proof as in Lemma 11 (i)). By Lemma 3*, $S_m(\hat{x}-) > 0$, hence $\sum_{k \geq i} \mathbf{q}(k|\cdot)$ is non-decreasing on $[0, \hat{x})$.

Note that it must be that $\sum_{k \geq i} \mathbf{q}(k|\cdot)$ is continuous at $\tilde{x}$. Indeed, suppose not. By Lemma 10 (iii), $S_m(\cdot) \sum_{k \geq i} \mathbf{q}(k|\cdot)$ is continuous at $\tilde{x}$. If $\sum_{k \geq i} \mathbf{q}(k|x)$ is discontinuous at $\tilde{x}$, it must be $\sum_{k \geq i} \mathbf{q}(k|\tilde{x}-) < 1$. But if $\sum_{k \geq i} \mathbf{q}(k|\tilde{x}) = 1$, then there are no sales made at location $\tilde{x}$ as $\mathbf{p}(\tilde{x}) > v^i$, and $S_m(\tilde{x}-) = S_m(\tilde{x}) > 0$ by Lemma 3*. Contradiction.

Using $\sum_{k \geq i} \mathbf{q}(k|x) S_m(x)$ is constant over $(0, \hat{x})$ again, we must have:

$$\begin{aligned}
\frac{1}{\sum_{k \geq i} \mathbf{q}(k|\tilde{x} - \Delta)} &= \frac{S_m(\tilde{x} - \Delta)}{S_m(\tilde{x})} = 1 + \frac{\int_{y \in (\tilde{x} - \Delta, \tilde{x}]} \rho_m(y) dD(y)}{S_m(\tilde{x})} \\
&\leq 1 + \frac{\int_{y \in (\tilde{x} - \Delta, \tilde{x})} (1 - \sum_{k \geq i} \mathbf{q}(k|y)) dD(y)}{S_m(\tilde{x})} \\
&\leq 1 + (1 - \sum_{k \geq i} \mathbf{q}(k|\tilde{x} - \Delta)) \frac{\int_{y \in (\tilde{x} - \Delta, \tilde{x})} dD(y)}{S_m(\tilde{x})}
\end{aligned}$$

where the first inequality is due to the fact quality indices above $i$ are not purchased on $(0, \hat{x})$, and the second inequality is due to the fact $\sum_{k \geq i} \mathbf{q}(k|\cdot)$ being non-decreasing on $(0, \hat{x})$. From the above, we then obtain:

$$\frac{1 - \sum_{k \geq i} \mathbf{q}(k|\tilde{x} - \Delta)}{\sum_{k \geq i} \mathbf{q}(k|\tilde{x} - \Delta)} \leq \frac{(1 - \sum_{k \geq i} \mathbf{q}(k|\tilde{x} - \Delta)) \int_{y \in (\tilde{x} - \Delta, \hat{x})} dD(y)}{S_m(\tilde{x})}$$

$$\frac{1}{\sum_{k\geq i}\mathbf{q}(k|\tilde{x}-\Delta)} \leq \frac{\int_{y\in(\tilde{x}-\Delta,\tilde{x})} dD(y)}{S_m(\tilde{x})}$$

where we used $\sum_{k\geq i}\mathbf{q}(k|\tilde{x}-\Delta) < 1$ for every $\Delta > 0$. Taking the limit as $\Delta \to 0$, the right-hand side is converging to 0. If the premise is true, the left-hand side must converge to 1 by continuity of $\sum_{k\geq i}\mathbf{q}(k|x)$ at $\tilde{x}$. We get a contradiction. The proof is analogous for the case where $\sum_{k\geq i}\mathbf{q}(k|x) < 1$ for all $x < \hat{x}$ but converges to 1.

□

**Lemma 13.** *Consider a market outcome $\mathbf{p} \in \mathcal{A}, D, \mathbf{q} \in \mathcal{E}_{\mathbf{p},\gamma}$ with positive sales. Let $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \leq v^i\}$, with a convention that $\hat{x}_i = 1$ whenever $\mathbf{p}(x) > v^i$ for all $x \in X$, and $x_i = 0$. Then,*

*(i) $\hat{x}_i$ is increasing in $i$*

*(ii) $\mathbf{p}(x) \in (v^{i+1}, v^i)$ $D$-a.s. on $[\hat{x}_i, \hat{x}_{i+1}]$*

*(iii) if $\int_{(0,\hat{x}^n)} dD(y) > 0$, then $\int_{\mathbf{p}(x)\geq v^n} dD(y) = 1$*

*Proof.* (i) The first part is straightforward from the fact $\inf\{x \in X : \mathbf{p}(x) \leq v^i\} \subseteq \inf\{x \in X : \mathbf{p}(x) \leq v^j\}$ for any $j \leq i$.

(ii) First, note that the statement is trivially true whenever $\mathbf{p}(x) > v^i$ on $X$ (in this case, $\hat{x}_i = 1$). Suppose now that $\hat{x}_i < 1$. Then, either $\mathbf{p}(\hat{x}_i) \leq v^i$, or for any $\Delta > 0$, there exists $x \in (\hat{x}, \hat{x} + \Delta)$ such that $\mathbf{p}(x) \leq v^i$.

I now establish that whenever $\mathbf{p}(\hat{x}_i) \leq v^i$ and $\sum_{k\geq i}\mathbf{q}(j|\hat{x}_i) > 0$, we must have $\mathbf{p} \leq v^i$ $D$-a.s. on $[\hat{x}_i, 1)$. Suppose the statement is false. Let $\tilde{x}_i$ denote the location where the statement is false for "the first time"

$$\tilde{x}_i = \sup\left\{y > \hat{x}_i : \int_{z\in(\hat{x}_i,y),\mathbf{p}(z)>v^i} dD(z) = 0\right\}$$

Suppose first that there is an atom at $\tilde{x}_i$ and the statement is false: $\delta(\tilde{x}_i) > 0$ and $\mathbf{p}(\tilde{x}_i) > v^i$. Then at $\tilde{x}_i$, there is a downward jump in $\mathbf{q}(j|\cdot)$ for all $j$ such that $v^j > \mathbf{p}(\tilde{x}_i)$ and $\mathbf{q}(j|\tilde{x}_i-) \in (0,1)$ (by Lemma 10 (iv)) and a downward jump in price. If $\rho_m(\tilde{x}_i) = 0$, then the consumer may receive a zero payoff at $\tilde{x}_i$. By Lemma 12, at $\hat{x}_i$ (or in the right-neighborhood of $\hat{x}_i$ by right-continuity of $\mathbf{q}$ by Lemma 10 (i)), a consumer may get a strictly positive payoff; hence, we obtain a contradiction with the optimality of the consumer's strategy. Alternatively, if $\rho_m(\tilde{x}_i-) > 0$, the consumer can achieve a strictly higher payoff at the locations that are in the left neighborhood of $\tilde{x}_i$. Again, we obtain a contradiction.

Suppose now $\delta(\tilde{x}_i) = 0$. Given the definition of $\tilde{x}_i$, for any $\Delta > 0$, $\int_{z\in([\tilde{x}_i,\tilde{x}_i+\Delta],\mathbf{p}(z)>v^i} dD(z) > 0$.

Note that we can make $\Delta$ small enough so that $\mathbf{p}(x) \in (v^i, v^{i-1})$ $D$-a.s. over $[\tilde{x}_i, \tilde{x}_i + \Delta]$. Indeed, by Lusin's Theorem, if $\mathbf{p}$ is Lebesgue-measurable, it coincides with a continuous function except possibly for a zero-measure set. By assumption, $\mathbf{p}(x) > v^i$ on a positive measure of $[\tilde{x}_i, \tilde{x}_i + \Delta]$, then it must be that $\mathbf{p}(x) > v^i$ a.e. on $[\tilde{x}_i, \tilde{x}_i + \Delta]$ for some $\Delta$. In addition, for small enough $\Delta$, consumer shops with zero probability at the locations where $\mathbf{p}(x) > v^{i-1}$ as $\mathbf{q}(i|\cdot)$ is right-continuous for every $i$ by Lemma 10 (i)

Hence, for small enough $\Delta$, $\mathbf{p}(y) \in (v^i, v^{i-1})$ $D$-a.s on $[\tilde{x}_i, \tilde{x}_i + \Delta]$, $D$ is continuous on $[\tilde{x}_i, \tilde{x}_i + \Delta]$ and the consumer shops with a positive probability at the locations inside $[\tilde{x}_i, \tilde{x}_i + \Delta]$ where he gets a playoff

$$\sum_{k=1}^{n} \mathbf{q}(k|x)(v^k - \mathbf{p}(x))_+ < \sum_{k<i} \mathbf{q}(k|x)(v^k - v^i) < \sum_{k<i} \mathbf{q}(k|\tilde{x}_i)(v^k - v^i)$$

where the quality composition of $\mathbf{q}(k|x)$ is decreasing over $[\tilde{x}_i, \tilde{x}_i + \Delta]$ for every $k < i$ by Lemma 11. Again, we obtain a contradiction with the optimality of the consumer's strategy.

(iii) Suppose not and let $\hat{x}_{n+1} = \inf\{x : \mathbf{p}(x) < v^n\}$. Suppose first that $\hat{x}_{n+1} = \hat{x}_n$. Let $\tilde{x}^h = \inf\{y \leq \hat{x}_n : \int_{z\ \in(y,\hat{x}_n)dD(y)=0}\}$. By Lemma 11, $\mathbf{q}$ remains constant over$(\tilde{x}^h, \hat{x}_{n+1}]$ for every $i$. There can be no atom at $\tilde{x}^h$, as at $\tilde{x}^h$, consumer's payoff is lower than that at $\hat{x}_n$:

$$\sum_{k=1}^{n} \mathbf{q}(k|\tilde{x}^h)(v^k - \mathbf{p}(\tilde{x}^h))_+ < \sum_{k=1}^{n} \mathbf{q}(k|\tilde{x}^h)(v^k - \mathbf{p}(\hat{x}_n)) = \sum_{k=1}^{n} \mathbf{q}(k|\hat{x}_n)(v^k - \mathbf{p}(\hat{x}_n))$$

where by the definition of $\hat{x}_n$: $\mathbf{p}(\tilde{x}^h) \geq v^n > \mathbf{p}(\hat{x}_{n+1}) = \mathbf{p}(\hat{x}_n)$.

Since there is no atom at $\tilde{x}^h$, Lemma 10 delivers $\mathbf{q}(i|\cdot)$ is continuous at $\tilde{x}^h$ for every $i$. But then, the consumer must shop with zero probability in a left neighborhood of $\tilde{x}^h$, as all these locations hold only marginally different quality composition but a discretely higher price compared to $\hat{x}_{n+1}$. But this is only possible if $\tilde{x}^h = 0$, which contradicts our assumption $\int_{(0,\hat{x}^n)} dD(y) > 0$.

Suppose now $\hat{x}_{n+1} > \hat{x}_n$. If $S_m(\hat{x}_{n+1}-) = 0$, the statement would be true. Conversely, suppose $S_m(\hat{x}_{n+1}-) > 0$. Then, by Lemma 11 (ii), $\mathbf{q}(i|\hat{x}_{n+1}-) = \mathbf{q}(i|\hat{x}_n)$ for every $i$. Again, we obtain a contradiction since either $\mathbf{q}$ remains constant over $(\tilde{x}^h, \hat{x}_{n+1})$, and the consumers suboptimally shop at high-priced locations; or consumers suboptimally shop at some outlet locations in $(\hat{x}_n, \hat{x}_{n+1})$ by paying a higher price for the same quality composition as in $\hat{x}_{n+1}$.

□

## G Proofs for the Binary Quality

In this section, I provide the results for a special case of a general model (see Appendix E) with a binary quality type.

*Proof of Lemma 2.* Follows from a more general Lemma 11. □

**Lemma 14.** *Consider a market outcome* $\mathbf{p} \in \mathcal{A}, (D, \mathbf{q}) \in \mathcal{E}_{\mathbf{p},\gamma}$ *with positive sales and the earliest outlet location* $\hat{x}$ *(*$\hat{x} = 1$ *if* $\mathbf{p}(x) > v^l$ *on* $X$*). Then, the total surplus is given by:*

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) = (D_o + \gamma)(1 - \mathbf{q}(\hat{x}-)) \left( \frac{\pi}{1-\pi} v^h + v^l \right) - \gamma \mathbf{q}(\hat{x}-)(v^h - v^l) - \gamma(\kappa + v^l)$$

*where* $D_o$ *is the mass of outlet shoppers:* $\int_{\mathbf{p}(x) \le v^l} dD(x) = D_o$.

*Proof.* By Lemma 13 (ii), all locations $[\hat{x}, 1)$ are outlet-locations $D$-a.s.. By Lemma 11 (ii), the quality composition remains constant over $[\hat{x}, 1)$ and coincides with $\mathbf{q}(\hat{x}-)$ $D$-a.s. (if $\hat{x} < 1$, and $\mathbf{q}$ is continuous at $\hat{x}$ by Lemma 10). Then, the total surplus is given by:

$$\begin{aligned} TS(\mathbf{p}, D, \mathbf{q}, \gamma) &= \int_0^{\hat{x}} v^h \mathbf{q}(x) dD(x) + D_o(\mathbf{q}(\hat{x}-) v^h + (1 - \mathbf{q}(\hat{x}-)) v^l) - \gamma\kappa \\ &= v^h \left( S_m(0) - S_m(\hat{x}-) \right) + D_o(\mathbf{q}(\hat{x}-) v^h + (1 - \mathbf{q}(\hat{x}-)) v^l) - \gamma\kappa \end{aligned}$$

Recall that $S_m(x)(1 - \mathbf{q}(x))$ is constant on $(0, \hat{x})$ by Lemma 11 (i), and $S_m(\hat{x}-) = D_o + \gamma$. Hence, we obtain:

$$\begin{aligned} TS(\mathbf{p}, D, \mathbf{q}, \gamma) &= (D_o + \gamma) v^h \left( \frac{1 - \mathbf{q}(\hat{x}-)}{1 - \pi} - 1 \right) + D_o(\mathbf{q}(\hat{x}-) v^h + (1 - \mathbf{q}(\hat{x})) v^l) - \gamma\kappa \\ &= (D_o + \gamma)(1 - \mathbf{q}(\hat{x}-)) \left( \frac{\pi}{1-\pi} v^h + v^l \right) - \gamma \mathbf{q}(\hat{x}-)(v^h - v^l) - \gamma(\kappa + v^l) \end{aligned}$$

□

*Proof of Lemma 4.* Follows from a more general Lemma 13. □

Consider some market outcome $\mathbf{p} \in \mathcal{A}, D, \mathbf{q} \in \mathcal{E}_{\mathbf{p},\gamma}$. Define $\hat{x} = \inf\{x : \mathbf{p}(x) \le v^l\}$. By Lemma 13, consumers only shop at outlet locations on $[\hat{x}, 1)$. Lemma 15 summarizes the bounds of screening for every possible mass of outlet shoppers: $D_o = \int_{\hat{x}}^1 dD(y)$.

*Proof of Lemma 5.* By Lemma 11, $\mathbf{q}$ is induced by $\mathbf{p}, \sigma$ on $[0, \hat{x}]$ if and only if $S_m(x)(1 - \mathbf{q}(x))$ remains constant over $[0, \hat{x}]$. If $D$ admits a density, then $S_m$ is differentiable almost everywhere on $X$. In particular, for almost every $x \in [0, \hat{x}]$:

$$S_m'(x) = -\sigma(x)\mathbf{q}(x)$$

Hence, $\mathbf{q}$ is also amost everywhere differentiable on $[0, \hat{x}]$, with a derivative:

$$\begin{aligned}
\mathbf{q}'(x) &= -\mathbf{q}(x)(1-\mathbf{q}(x))\frac{\sigma(x)}{S_m(x)} \\
&= -\mathbf{q}(x)(1-\mathbf{q}(x))^2\frac{\sigma(x)}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))}
\end{aligned}$$

Then, from the above, we can solve out for the cumulative number of shoppers at all locations below $x$ for any $x < \hat{x}$ such that $\mathbf{q}(x) > 0$—which holds everywhere on $[0, \hat{x}]$ by Lemma 12. We obtain that

$$\begin{aligned}
\frac{\int_0^x \sigma(y)dy}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))} &= \int_0^x -\frac{\mathbf{q}'(y)}{\mathbf{q}(y)(1-\mathbf{q}(y))^2}dx \\
&= \int_{\mathbf{q}(x)}^{\pi} \frac{1}{q(1-q)^2}dq = \ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(x)}{\mathbf{q}(x)}\right) + \frac{\pi}{1-\pi} - \frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}
\end{aligned}$$

Rearranging, we get:

$$\begin{aligned}
\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} + \ln\left(\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}\right) &= \ln\left(\frac{\pi}{1-\pi}\right) + \frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))} \\
\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}\exp\left[\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}\right] &= \frac{\pi}{1-\pi}\exp\left[\frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))}\right] \\
\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} &= W\left(\frac{\pi}{1-\pi}\exp\left[\frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))}\right]\right)
\end{aligned}$$

In particular, the above is true at $\mathbf{q}(\hat{x})$ whenever:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right) = \frac{1}{(1-\mathbf{q}(\hat{x}))S_m(\hat{x})} - \frac{1}{1-\pi}$$

If $\hat{x}$ is an outlet threshold, then $S_m(\hat{x}) = \int_{\hat{x}}^1 \sigma(y)dy$, as all locations in $[\hat{x}, 1)$ are outlet $\sigma$-a.s., delivering Equation (Q-T).

□

*Proof of Proposition 4.* Take any $q$. I now construct a $\hat{x}$-threshold market outcome $\mathbf{p} \in \mathcal{A}, (\mathbf{q}, \sigma) = \mathcal{E}_{\mathbf{p}}$ with $\mathbf{q}(\hat{x}) = q$. Take $\sigma(x) = 1, \forall x \in X$.

Let $\hat{x}$ be as suggested by Equation (Q-T):

$$(1-q)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+\frac{1}{1-\pi}\right)=\frac{1}{1-\hat{x}}$$

Note that for any $q \in (0,\pi]$, $\hat{x} \in [0,1)$. We let the prices be low for all locations in $[\hat{x},1]$: $\mathbf{p}(x)=v^l, \forall x \geq \hat{x}$. Define the quality composition to be $\mathbf{q}(x)=q, \forall x \geq \hat{x}$. By Lemma 2, $\mathbf{q}$ is induced by $(\sigma,\mathbf{p})$ on $[\hat{x},1]$.

For earlier locations, define $\mathbf{q}$ so that:

$$\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}=W\left(\frac{\pi}{1-\pi}\exp\left[\frac{\pi}{1-\pi}-\frac{x}{(1-\hat{x})q}\right]\right)$$

for all $x \in [0,\hat{x}]$. By Lemma 5, $\mathbf{q}$ is induced by $(\sigma,\mathbf{p})$ on $[0,\hat{x}]$. Then, $\mathbf{q}$ is induced by $(\sigma,\mathbf{p})$.

Finally, complete the construction by letting $\mathbf{p}(x)=v^h-\frac{q}{\mathbf{q}(x)}(v^h-v^l), \forall x \in [0,\hat{x})$. Then, consumers are indifferent between all locations, and $\sigma$ is optimal given $(\mathbf{p},\mathbf{q})$. In addition, $\mathbf{p} \in \mathcal{A}$ whenever $\hat{x}<1$.

□

**Lemma 15.** *Consider a market outcome $\mathbf{p} \in \mathcal{A}, D, \mathbf{q} \in \mathcal{E}_{\mathbf{p},\gamma}$ with positive sales. Then, the quality composition at the earliest outlet location $q^o$ and the mass of buyers shopping at outlet locations $D_o=\int_{\hat{x}}^1 dD(y)$ satisfy the following. For every $D_o \leq 1$ and $D_o+\gamma>0$, $q^o \in [\underline{q}^o,\bar{q}^o]$, with:*

$$1+\gamma=(D_o+\gamma)(1-\underline{q}^o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-\underline{q}^o}{\underline{q}^o}\right)+\frac{1}{1-\pi}\right]$$
$$\bar{q}^o=\pi\frac{D_o+\gamma}{D_o+\gamma+(1-D_o)(1-\pi)}$$

*In addition:*

*i) If under $D$, the mass of shoppers at outlet locations is $D_o$, and $D$ admits no atoms at non-outlet locations, then the quality composition at (almost all) outlet locations is $\underline{q}^o$.*

*ii) If there exists a unique non-outlet location that attracts a mass $1-D_o$ of consumers, the quality composition at (almost all) outlet locations is $\bar{q}^o$.*

*iii) If there are finitely many non-outlet locations, where $D$ is discontinuous, then $q^o>\underline{q}^o$.*

*Proof.* **Step 1**. First, note that the statement is true for $D_o=1$, since all consumers shop at the outlet locations. The steady-state condition requires the quality composition to be constant across the locations, with $\tilde{\mathbf{q}}(x)=\pi$. The above is satisfied.

**Step 2.1**. Now suppose that $-\gamma < D_o < 1$. Then, by Lemma 4, $\mathbf{p}(x) = v^l$ $D$-a.s. on $[\hat{x}, 1]$, that is for every $x \in [\hat{x}, 1]$, $S_m(x) = 1 - D(x) + \gamma$. By Lemma 3*, $S_m(\hat{x}) > 0$ whenever the market outcome $m = \langle \mathbf{p}, D, \mathbf{q}, \gamma \rangle$ admits positive sales.

By Lemma 11 $\mathbf{q}$ is induced over $[\hat{x}, 1]$ by $(D, \mathbf{p}, \gamma)$ if and only if it remains constant over $[\hat{x}, 1]$ for almost all such locations (except possibly for the location at the right boundary of the support of the consumer's strategy). By continuity of $\mathbf{q}$ at $\hat{x}$ due to Lemma 10, $\mathbf{q}(\hat{x}) = q^o$.

Then, by Lemma 5 $\mathbf{q}$ is induced by $(D, \mathbf{p}, \gamma)$ over $[0, \hat{x}]$ if and only if:

$$\frac{q^o}{1 - q^o} = W\left(\frac{\pi}{1 - \pi} \exp\left[\frac{\pi}{1 - \pi} - \frac{D(\hat{x})}{(1 - q^o) S_m(\hat{x})}\right]\right)$$

As all locations below $\hat{x}$ are outlet locations almost surely, then $S_m(\hat{x}) = \int_{\hat{x}}^1 dD(y) + \gamma$. Rewriting the above equation, we obtain:

$$1 + \gamma = (D_o + \gamma)(1 - q^o)\left[\ln\left(\frac{\pi}{1 - \pi} \frac{1 - q^o}{q^o}\right) + \frac{1}{1 - \pi}\right]$$

Note that every shopping strategy that is absolutely continuous on an interval containing all non-outlet locations induces the same outlet quality composition $\underline{q}^o$ for a given mass of outlet shoppers $D_o$.

**Step 2.2**. I show that for any two market outcomes with the same prices $\mathbf{p}$, shopping strategy $D$, and disposal rate $\gamma$, the outlet quality composition $q^o$ is unique.

Suppose not and there exist $m = \langle \mathbf{p}, D, \mathbf{q}, \gamma \rangle$ and $\tilde{m} = \langle \mathbf{p}, D, \tilde{\mathbf{q}}, \gamma \rangle$ with $\mathbf{p} \in A$ and $(D, \mathbf{q}), (D, \tilde{\mathbf{q}}) \in \mathcal{E}_{\mathbf{p},\gamma}$, such that $\tilde{\mathbf{q}}(\hat{x}) = \tilde{q}_o < q^o = \mathbf{q}(\hat{x})$ (the other case is symmetric).

By Lemma 10, $\mathbf{q}$ is continuous at $\hat{x}$. Hence, there exists a left neighborhood of $\hat{x}$, such that $\mathbf{q}(\cdot) > \tilde{\mathbf{q}}(\cdot)$ for all stores within such neighborhood. Let $x_1 = \inf\{x < \hat{x} : \mathbf{q}(x) > \tilde{\mathbf{q}}(x)\}$. Note that in this case, it must be that at $x_1$, the total sales volume is higher in the first market outcome than in the second one due to its superior quality composition on $[x_1, \hat{x}]$:

$$S_{\tilde{m}}(x_1) = D_o + \gamma + \int_{x_1}^{\hat{x}} \tilde{\mathbf{q}}(x) dD(x) < D_o + \gamma + \int_{x_1}^{\hat{x}} \mathbf{q}(x) dD(x) = S_m(x_1)$$

By Lemma 11, $\mathbf{q}$ and $\tilde{\mathbf{q}}$ being induced by $(D, \mathbf{p}, \gamma)$ requires:

$$(1 - \tilde{\mathbf{q}}(x_1)) = (1 - \tilde{q}_o)\frac{D_o + \gamma}{S_{\tilde{m}}(x_1)} > (1 - q^o)\frac{D_o + \gamma}{S_m(x_1)} = (1 - \mathbf{q}(x_1))$$

Hence, $\tilde{\mathbf{q}}(x_1) < \mathbf{q}(x_1)$. In addition, by the same reasoning $\tilde{\mathbf{q}}(x_1-) < \mathbf{q}(x_1-)$. Then, it must be that $x_1 = 0$ (or else $x_1$ is not correctly defined) and $\pi = \mathbf{q}(0) > \tilde{\mathbf{q}}(0) = \pi$, we obtain a contradiction.

**Step 2.3**. Whenever $D$ is continuous, we can approximate it with some sequence of absolutely continuous $D^n$. For these, we know how to construct an induced steady-state from the previous step. I now go over this formally. I will show that if $D$ is continuous on $[x_1, x_2]$, then:

$$\frac{\mathbf{q}(x_2)}{1-\mathbf{q}(x_2)} = W\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\exp\left[\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)} - \frac{D(x_1)-D(x_2)}{(D_o+\gamma)(1-q^o)}\right]\right)$$

Consider a sequence of shopping strategies such that $D^n$ admits a density almost everywhere on $[x_1, x_2]$ (for instance, take $D^n$ to be piece-wise uniform) that pointwise converges to $D(x)$ on $[x_1, x_2]$.

Construct $\mathbf{q}^n(x)$ so that $(S_m(x_1) - \int_{x_1}^x \mathbf{q}^n(x)dD^n(y))(1-\mathbf{q}^n(x))$ remains constant on $[x_1, x_2]$ and equals $(1-q^o)(D_o+\gamma)$. For each $D^n$, $\mathbf{q}^n(x)$ is itself absolutely continuous and is given by:

$$\frac{\mathbf{q}^n(x)}{1-\mathbf{q}^n(x)} = W\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\exp\left[\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)} - \frac{D^n(x)-D^n(x_2)}{(D_o+\gamma)(1-q^o)}\right]\right)$$

In addition, since $D^n$ converges to $D$ on $[x_1, x_2]$, then $\mathbf{q}^n(x)$ converges (pointwisely) to $\tilde{\mathbf{q}}$ with:

$$\frac{\tilde{\mathbf{q}}(x)}{1-\tilde{\mathbf{q}}(x)} = W\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\exp\left[\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)} - \frac{D(x)-D(x_2)}{(D_o+\gamma)(1-q^o)}\right]\right)$$

I now show that $(1-\tilde{\mathbf{q}}(x))(S_m(x_1) - \int_{x_1}^x \tilde{\mathbf{q}}(y)dD(y))$ remains constant over $[x_1, x_2]$ and equals $(1-q^o)(D_o+\gamma)$. By construction, we have:

$$\left(S_m(x_1) - \int_{x_1}^x \mathbf{q}^n(y)dD^n(y)\right)(1-\mathbf{q}^n(x)) = (1-q^o)(D_o+\gamma)$$

Hence, to establish the result, it suffices to show that $\left(S_m(x_1) - \int_{x_1}^x \mathbf{q}^n(y)dD^n(y)\right)(1 - \mathbf{q}^n(x))$ converges to $(1-\tilde{\mathbf{q}}(x))(S_m(x_1) - \int_{x_1}^x \tilde{\mathbf{q}}(y)dD(y))$ for every $x$. Since $[x_1, x_2]$ is compact, $D$ is uniformly continuous on $[x_1, x_2]$ by Heine–Cantor theorem. Hence, $D^n$ converges to $D$ uniformly. Since $\mathbf{q}(x_1)$ is bounded by $\pi$, and $D_o+\gamma > 0$ in any market outcome with positive sales, the argument inside $W$ is bounded. Then, $W$ is uniformly continuous on $[0, K]$ for some $K$ large enough and $\mathbf{q}^n(x)$ converges uniformly to $\tilde{\mathbf{q}}$.

It remains to verify that $\int_{x_1}^{x} \mathbf{q}^n(y)dD^n(y)$ converges to $\int_{x_1}^{x} \tilde{\mathbf{q}}(y)dD(y)$ for every $x$:

$$\int_{x_1}^{x} \mathbf{q}^n(y)dD^n(y) = \int_{x_1}^{x} \tilde{\mathbf{q}}(y)dD^n(y) + \int_{x_1}^{x} \mathbf{q}^n(y) - \tilde{\mathbf{q}}(y)dD^n(y)$$

by Portmanteau theorem, $\int_{x_1}^{x} \tilde{\mathbf{q}}(y)dD^n(y)$ converges $\int_{x_1}^{x} \tilde{\mathbf{q}}(y)dD(y)$, as $\tilde{\mathbf{q}}$ is continuous on $[x_1, x_2]$. Hence, it now only remains to show that $\int_{x_1}^{x} \mathbf{q}^n(y) - \tilde{\mathbf{q}}(y)dD^n(y)$ converges to 0. As $\mathbf{q}^n(y)$ converges to $\tilde{\mathbf{q}}(y)$ uniformly, then for any $\varepsilon > 0$ for sufficiently large $n$:

$$\varepsilon \geq \varepsilon \int_{x_1}^{x} dD^n(y) \geq \int_{x_1}^{x} \mathbf{q}^n(y) - \tilde{\mathbf{q}}(y)dD^n(y) \geq -\varepsilon \int_{x_1}^{x} dD^n(y) \geq -\varepsilon$$

Taking $\varepsilon \to 0$, we get the desired convergence.

Suppose now that $\mathbf{q}(x_2) > \tilde{\mathbf{q}}(x_2)$ (the other case is symmetric). We can obtain contradiction in a similar fashion as in the previous step. Since $\mathbf{q}$ is continuous on $[x_1, x_2]$ by Lemma 10, and $\tilde{\mathbf{q}}$ is continuous by construction, there exists $\tilde{x} \in [x_1, x_2)$ such that $\mathbf{q}(\tilde{x}) = \tilde{\mathbf{q}}(\tilde{x})$ but $\mathbf{q}(x) > \tilde{\mathbf{q}}(x)$ for all $x \in (\tilde{x}, x_2]$. Then, as $\mathbf{q}$ is induced by $(D, \gamma, \mathbf{p})$, and by construction of $\tilde{\mathbf{q}}(x)$ we must have: $S_m(\tilde{x}) = S_m(x_1) - \int_{x_1}^{x} \tilde{\mathbf{q}}(x)dD(y)$. Moreover, from Lemma 2 (i):

$$\begin{aligned} 1 - \mathbf{q}(x_2) = \frac{(D_o + \gamma)(1 - q_o)}{S_m(x_2)} = (1 - \mathbf{q}(\tilde{x}))\frac{S_m(\tilde{x})}{S_m(x_2)} &= (1 - \mathbf{q}(\tilde{x}))\frac{S_m(\tilde{x})}{S_m(\tilde{x}) - \int_{\tilde{x}}^{x_2} \mathbf{q}(x)dD(x)} \\ &> \frac{S_m(\tilde{x})}{S_m(\tilde{x}) - \int_{\tilde{x}}^{x_2} \tilde{\mathbf{q}}(x)dD(x)} = 1 - \tilde{\mathbf{q}}(x_2) \end{aligned}$$

which contradicts the premise of $\mathbf{q}(x_2) > \tilde{\mathbf{q}}(x_2)$.

Consequently, if $D$ has no atoms on $(0, \hat{x})$, then $\mathbf{q}(\hat{x}-) = \underline{q}^o$. By Lemma 10, $\mathbf{q}$ is continuous at $\hat{x}$, and $\mathbf{q}(\hat{x}) = \underline{q}^o$.

**Step 3**. Finally, let me show that the quality composition is within the suggested boundaries for every shopping strategy $D$. Note that if at $\tilde{x} \in X$ $D$ admits an atom, then using Lemma 10 (iii):

$$\begin{aligned} \frac{\mathbf{q}(\tilde{x}-) - \mathbf{q}(\tilde{x})}{(1 - \mathbf{q}(\tilde{x}-))} &= \frac{\delta(\tilde{x})\mathbf{q}(\tilde{x})}{S_m(\tilde{x})} \\ \frac{\mathbf{q}(\tilde{x}-) - \mathbf{q}(\tilde{x})}{(1 - \mathbf{q}(\tilde{x}-))\mathbf{q}(\tilde{x})} &= \frac{\delta(\tilde{x})}{S_m(\tilde{x})} \end{aligned} \tag{8}$$

First, suppose the seller has a unique non-outlet location $\tilde{x} \in (0, \hat{x})$. Then, $\delta(\tilde{x}) = 1 - D_o$.

The steady-state condition requires:

$$\begin{aligned}(1-\mathbf{q}(0))S_m(0) &= (1-\mathbf{q}(\tilde{x}))S_m(\tilde{x})\\ (1-\pi)\left(\mathbf{q}(\tilde{x})(1-D_o)+D_o+\gamma\right) &= (1-\mathbf{q}(\tilde{x}))(D_o+\gamma)\end{aligned}$$

Solving out for $\mathbf{q}(\tilde{x})$, we get:

$$\mathbf{q}(\tilde{x}) = \pi\frac{D_o+\gamma}{D_o+\gamma+(1-D_o)(1-\pi)}$$

Since all locations on $(\tilde{x},1]$ are $D$-a.s. outlet locations, we get that $q^o = \mathbf{q}(\tilde{x})$, and $q^o$ achieves the upper bound from the formulation of the lemma.

Now, consider any shopping strategy $D$. Then, for every two non-outlet locations $x_1, x_2 < \hat{x}$:

$$\frac{\mathbf{q}(x_1)-\mathbf{q}(x_2)}{(1-\mathbf{q}(x_1))} = \frac{\int_{y\in(x_1,x_2]}\mathbf{q}(y)dD(y)}{S_m(x_2)} \geq \frac{\mathbf{q}(x_2)\left(D(x_2)-D(x_1)\right)}{S_m(x_2)}$$

In particular, taking $x_1 = 0$ and $x_2 \to \hat{x}-$, we obtain:

$$\begin{aligned}\frac{\pi-\mathbf{q}(\hat{x}-)}{1-\pi} &\geq \frac{\mathbf{q}(\hat{x}-)(1-D_o)}{D_o+\gamma}\\ \mathbf{q}(\hat{x}-) &\leq \pi\frac{D_o+\gamma}{D_o+\gamma+(1-\pi)(1-D_o)}\end{aligned}$$

Since $\mathbf{q}$ is continuous at $\hat{x}$, we get that $q^o = \mathbf{q}(\hat{x}) = \mathbf{q}(\hat{x}-)$ is at most $\bar{q}^o$.

By the previous step, whenever $D$ is continuous on an interval $[x_1, x_2)$, steady-state quality composition $\mathbf{q}$ satisfies:

$$\ln\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\frac{1-\mathbf{q}(x_2-)}{\mathbf{q}(x_2-)}\right)+\frac{1}{1-\mathbf{q}(x_1)}-\frac{1}{1-\mathbf{q}(x_2-)} = \frac{D(x_2-)-D(x_1)}{(D_o+\gamma)(1-q^o)}$$

if there is a jump at $x_2$, then by Equation (8) and $(D_o+\gamma)(1-q^o) = S_m(x_2)(1-\mathbf{q}(x_2))$ by Lemma 11, we get that the overall change over an interval $[x_1, x_2]$ is:

$$\begin{aligned}&\ln\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\frac{1-\mathbf{q}(x_2-)}{\mathbf{q}(x_2-)}\right)+\frac{1}{1-\mathbf{q}(x_1)}-\frac{1}{1-\mathbf{q}(x_2-)}\\ &+\frac{\mathbf{q}(x_2-)-\mathbf{q}(x_2)}{(1-\mathbf{q}(x_2-))\mathbf{q}(x_2)}\frac{1}{1-\mathbf{q}(x_2)} = \frac{D(x_2)-D(x_1)}{(D_o+\gamma)(1-q^o)}\end{aligned} \tag{9}$$

Since ln is (strictly) concave, it satisfies for any $y > 0$: $\ln(y) \leq y - 1$, with a strict inequality for any $y \neq 1$. In particular, we obtain the following bound:

$$\begin{aligned}
&\ln\left(\frac{\mathbf{q}(x_2-)}{1-\mathbf{q}(x_2-)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)}\right) \leq \frac{\mathbf{q}(x_2-)}{1-\mathbf{q}(x_2-)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)} - 1\\
&\Rightarrow \ln\left(\frac{\mathbf{q}(x_2-)}{1-\mathbf{q}(x_2-)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)}\right) + \frac{1}{1-\mathbf{q}(x_2-)} - \frac{1}{1-\mathbf{q}(x_2)}\\
&\leq \frac{\mathbf{q}(x_2-)}{1-\mathbf{q}(x_2-)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)} - 1 + \frac{1}{1-\mathbf{q}(x_2-)} - \frac{1}{1-\mathbf{q}(x_2)}\\
&= \frac{\mathbf{q}(x_2-)-\mathbf{q}(x_2)}{(1-\mathbf{q}(x_2-))\mathbf{q}(x_2)}\frac{1}{1-\mathbf{q}(x_2)}
\end{aligned}$$

Plugging this back into Equation (9), we obtain that over $[x_1, x_2]$, the change in $\mathbf{q}$ is given by at most:

$$\ln\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)}\right) + \frac{1}{1-\mathbf{q}(x_1)} - \frac{1}{1-\mathbf{q}(x_2)} \leq \frac{D(x_2)-D(x_1)}{(D_o+\gamma)(1-q^o)}$$

In addition, the inequality is strict if there is a discontinuity of $D$ at $x_2$. Since the above is true for every interval, it must be that:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o}{q^o}\right) + \frac{1}{1-\pi} - \frac{1}{1-q^o} \leq \frac{1-D_o}{(D_o+\gamma)(1-q^o)}$$

and $q^o$ is at at least $\underline{q}^o$.

For finitely many discontinuities, we can find $\{x_1, \ldots, x_n\}$ of non-outlet locations, such that $D$ is discontinuous at $x_i$. Summing over $[0, x_1], \ldots [x_i, x_{i+1}], [x_n, \hat{x}]$, we obtain:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o}{q^o}\right) + \frac{1}{1-\pi} - \frac{1}{1-q^o} < \frac{1-D_o}{(D_o+\gamma)(1-q^o)}$$

Part (iii) follows.

□

# H Omitted Proofs for Section 3.2

In this appendix, I analyze the properties of the seller's payoff as a function of the outlet quality composition $\tilde{V}^S$.

**Lemma 16.** $\tilde{V}^S(\cdot)$ *has the following properties:*

*i)* $\tilde{V}^S(\pi) = v^l$

*ii)* $\frac{\partial \tilde{V}^S}{\partial q}(\pi) > 0$

*iii)* $\lim_{q\to 0}\tilde{V}^S(q) = 0$, $\lim_{q\to 0}\frac{\partial \tilde{V}^S(q)}{\partial q} = \infty$

*iv)* $\tilde{V}^S$ *is concave-convex: that is, there exists* $\bar{q}(\pi) \in (0, \pi/2)$, *such that* $\tilde{V}^S$ *is convex on* $(\bar{q}(\pi), \pi]$ *and is concave on* $[0, \bar{q}(\pi))$

*Proof.*

$$\frac{\partial \tilde{V}^S}{\partial q} = \left(\frac{\pi}{1-\pi}v^h + v^l\right) / \left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right)^2 \frac{1}{(1-q)q} - (v^h - v^l)$$

i) Is straightforward from plugging in $q = \pi$.

ii) Consider $\frac{\partial \tilde{V}^S}{\partial q}(\pi)$:

$$\frac{\partial \tilde{V}^S}{\partial q}(\pi) = \left(\frac{\pi}{1-\pi}v^h + v^l\right)\frac{1-\pi}{\pi} - (v^h - v^l) = \frac{v^l}{\pi} > 0$$

iii)

$$\lim_{q\to 0}\tilde{V}^S \propto \lim_{q\to 0}\frac{1}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)(1-\pi) + 1} = 0$$

$$\lim_{q\to 0}\frac{\partial \tilde{V}^S(q)}{\partial q} \propto \lim_{q\to 0}\frac{1/[(1-q)q]}{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right)^2}$$

Applying L'Hôpital's rule twice, we can compute the above limit as:

$$\lim_{q\to 0}\frac{1/[(1-q)q]}{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right)^2} = \lim_{q\to 0} 2\frac{(1-2q)/[(1-q)q]}{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right)} = \lim_{q\to 0} 2\frac{-1+2q-2q^2}{(1-q)q} = \infty$$

iv)

$$\frac{\partial^2 \tilde{V}^S}{\partial q^2} / \left(\frac{\pi}{1-\pi}v^h + v^l\right) =$$
$$\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right)^{-3}((1-q)q)^{-2}\left(2 - (1-2q)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right)\right)$$

Whenever $q \geq \min\{1/2, \pi\}$, the above is positive. For $q < 1/2$, the expression in parentheses increases with $q$, and is negative at $q \to 0$. Hence, it must be that there is a unique threshold $\bar{q}(\pi) \in (0, \min\{\pi, 0.5\})$, such that the second derivative is positive for $q > \bar{q}(\pi)$ but is negative for $q < \bar{q}(\pi)$. In addition,

$$2 = (1 - 2\bar{q}(\pi)) \left( \ln \left( \frac{\pi}{1-\pi} \frac{1-\bar{q}(\pi)}{\bar{q}(\pi)} \right) + \frac{1}{1-\pi} \right) \leq (1 - 2\bar{q}(\pi)) \frac{\pi}{1-\pi} \frac{1}{\bar{q}(\pi)}$$
$$\bar{q}(\pi) \leq 0.5\pi$$

where I used again the boundary on $\ln(y) \leq y - 1$ for all $y > 0$.

With some abuse of notation, let $\tilde{V}^S : (0, 1] \times [0, 1] \times \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ denote:

$$\tilde{V}^S(q, \pi, v^h, v^l) = \left(\pi v^h + (1-\pi) v^l\right) / \left( \ln \left( \frac{\pi}{1-\pi} \frac{1-q}{q} \right) (1-\pi) + 1 \right) - q(v^h - v^l)$$

□

*Proof of* **??**. Since $\tilde{V}^S(\cdot, \pi, v^h)$ is concave-convex by Lemma 16, at most, two points satisfy FOC in $q$ for every $\pi, v^h$, and the only solution candidate is the minimum of such points (as, otherwise, the solution candidate fails to satisfy SOC). Moreover, $\tilde{V}^S(q, \pi, v^h, v^l) \leq \tilde{V}^S(\pi, \pi, v^h, v^l)$ for all $q \in [\bar{q}(\pi), \pi]$, as $\tilde{V}^S$ is convex in $q$ on this interval.

The interior solution candidate $q_a^o : [0, 1] \times \mathbb{R}_{++} \to [0, 1]$ can be implicitly defined as :

$$q_a^o(\pi, v^h, v^l) = \min \left\{ q \in [0, \pi] : \frac{\partial \tilde{V}^S(q, \pi, v^h, v^l)}{\partial q} = 0 \right\}$$

with a convention that $q_a^*(\pi, v^h, v^l) = \pi$ if the set $\min \left\{ q \in [0, \pi] : \frac{\partial \tilde{V}^S(q, \pi, v^h, v^l)}{\partial q} = 0 \right\}$ is empty. By Lemma 16, $\frac{\partial \tilde{V}^S(q, \pi, v^h, v^l)}{\partial q}$ is positive both at $q = \pi$ and as $q \to 0$. If $q_a^*(\pi, v^h, v^l) = \pi$, then $\frac{\partial \tilde{V}^S(q, \pi, v^h, v^l)}{\partial q} \geq 0$ for all $q$, and $\tilde{V}^S(\cdot, \pi, v^h, v^l) \leq \tilde{V}^S(\pi, \pi, v^h, v^l)$.

If $q_a^*(\pi, v^h, v^l) \neq \pi$, then $\tilde{V}^S(q, \pi, v^h, v^l) \leq \tilde{V}^S(q_a^*(\pi, v^h, v^l), \pi, v^h, v^l)$ for all $q \in (0, \bar{q}(\pi)]$, as $\tilde{V}^S$ is concave in $q$ on this interval. The result follows.

□

**Proposition 9.** *There exist $\bar{\pi}(v^h, v^l)$ and $\bar{v}^h(\pi, v^l)$, such that $\tilde{V}^S$ attains (does not attain) its maximum at $q = \pi$ if either $\pi < \bar{\pi}(v^h, v^l)$ or $v^h < \bar{v}^h(\pi, v^l)$.*

*Proof.* **Step 1**. First, I establish that the seller engages in no screening at $\pi \to 0$ or $v^h \to v^l$; but engages in active screening for $\pi \to 1$ or $v^h \to \infty$.

Note that we have boundary on $\tilde{V}^S(\cdot, \pi, v^h, v^l)$:

$$\tilde{V}^S(q, \pi, v^h, v^l) \leq \pi v_h + (1-\pi)v^l - q(v^h - v^l)$$

with a strict inequality for any market outcome with positive screening $q < \pi$ (due to positive screening distortion). As $\pi \to 0$ or $v^h \to v^l$, the boundary converges to $v^l$ for every feasible $q$, and the seller strictly prefers to do no screening.

$\pi \to 1$: Suppose the seller induces $q^o = \frac{1-\pi}{\pi}$ which is feasible for sufficiently large $\pi$. In addition, $\frac{1-\pi}{\pi} < \bar{q}(\pi)$ for large enough $\pi$. Hence, as $\pi \to 1$, we obtain the following boundary on the seller's payoff from an interior solution:

$$\tilde{V}^S\left(q_a^o(\pi, v^h, v^l), \pi, v^h, v^l\right) \geq \tilde{V}^S\left(\frac{1-\pi}{\pi}, \pi, v^h\right) = \left(\pi v^h + (1-\pi)v^l\right) / \left(2\ln\left(\frac{\pi}{1-\pi}\right)(1-\pi) + 1\right) - \frac{1-\pi}{\pi}(v^h - v^l) \underset{\pi \to 1}{\to} v^h > v^l$$

$v^h \to \infty$: Similarly, fix any $q < \bar{q}(\pi)$. Then, the seller's payoff from the interior solution is at least:

$$\tilde{V}^S\left(q_a^o(\pi, v^h, v^l), \pi, v^h, v^l\right) \geq \tilde{V}^S(q, \pi, v^h) = (v^h - v^l)\left[\frac{\frac{\pi}{1-\pi}v^h + v^l}{v^h - v^l} / \left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right) - q\right] \underset{v^h \to \infty}{\to} \infty$$

To verify this, it is enough to establish that the expression under the square brackets is bounded by some positive constant for every $v^h$. Indeed, note first that for any for any $v^h > v^l > 0$:

$$\frac{\frac{\pi}{1-\pi}v^h + v^l}{v^h - v^l} > \frac{\pi}{1-\pi}$$

Bounding $\ln(\cdot)$ by $\ln(y) < y - 1$ for every $y > 1$:

$$\frac{\frac{\pi}{1-\pi}v^h + v^l}{v^h - v^l} / \left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right) - q > \frac{\pi}{1-\pi}\frac{1}{\frac{\pi}{1-\pi}\frac{1-q}{q} + \frac{\pi}{1-\pi}} - q = 0$$

Hence, some positive constant $c > 0$ exists, such that the squared expression is strictly above $c$ for every $v^h$. This completes Step 1.

**Step 2**. Now, I show that switching between the two solution types can only happen once. Let

$$\bar{\pi}(v^h, v^l) = \inf_{\pi \geq \hat{\pi}(v^h, v^l)} \left\{\tilde{V}^S(q_a^o(\pi, v^h, v^l), \pi, v^h, v^l) > v^l\right\}$$

To avoid notational complications, when analyzing comparative statics with respect to $\pi$, I drop $v^h$ and $v^l$ as arguments for $\bar{\pi}$, $q_a^o$ or $\tilde{V}^S$.

Note that it must be that $q_a^o(\bar{\pi}) \neq \bar{\pi}$ and $\frac{d\tilde{V}^S}{d\pi}(q_a^o(\bar{\pi}), \bar{\pi}) > 0$, as otherwise there can be no switch at $\bar{\pi}$ between the two solutions. I establish now that for any $\pi > \bar{\pi}$, $\frac{d\tilde{V}^S}{d\pi}(q_a^o(\pi), \pi) > 0$.

Given $q_a^o(\pi, v^h, v^l)$ satisfies FOC with respect to $q$:

$$\begin{aligned}\frac{d\tilde{V}^S}{d\pi}(q_a^o(\pi), \pi) &= \frac{\partial \tilde{V}^S}{\partial \pi}(q_a^o(\pi), \pi) \\ &= \frac{1}{(1-\pi)^2} \frac{v^h}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}} - \frac{\frac{\pi}{1-\pi}v^h + v^l}{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}\right)^2} \frac{1}{\pi(1-\pi)^2}\end{aligned}$$

Hence, the sign of $\frac{d\tilde{V}^S}{d\pi}(q_a^o(\pi), \pi)$ coincides with the sign of:

$$\begin{aligned}F(\pi) &\equiv \pi v^h - \frac{\frac{\pi}{1-\pi}v^h + v^l}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}} \\ \frac{dF}{d\pi} &= v^h - \frac{1}{(1-\pi)^2}\frac{v^h}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}} \\ &+ \frac{\frac{\pi}{1-\pi}v^h + v^l}{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}\right)^2}\left(\frac{1}{\pi(1-\pi)^2} - \frac{1}{q_a^o(\pi)(1-q_a^o(\pi))}\frac{dq_a^o}{d\pi}\right) \\ &= v^h - F(\pi)\frac{1}{(1-\pi)^2}\frac{1}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}} \\ &- \frac{\frac{\pi}{1-\pi}v^h + v^l}{\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(\pi)}{q_a^o(\pi)}\right) + \frac{1}{1-\pi}\right)^2}\frac{1}{q_a^o(\pi)(1-q_a^o(\pi))}\frac{\partial q_a^o}{\partial \pi}\end{aligned}$$

Since $\tilde{V}^S$ is concave in $q^o$ at $q_a^o(\pi)$, the sign of $\frac{\partial q_a^o}{\partial \pi}$ is determined by the sign of $\frac{\partial^2 \tilde{V}^S}{\partial \pi \partial q^o}(q_a^o(\pi), \pi)$. Suppose the premise is wrong, then there exists some $\tilde{\pi} > \bar{\pi}$, such that $F(\tilde{\pi}) = 0$ and $\frac{dF}{d\pi}(\tilde{\pi}) < 0$. However, $\frac{\partial^2 \tilde{V}^S}{\partial q^o \partial \pi} < 0$ whenever $F(\tilde{\pi}) \leq 0$. Then, $\frac{\partial q_a^o}{\partial \pi}(\tilde{\pi}) < 0$ and hence $\frac{dF}{d\pi}(\tilde{\pi}) > 0$. We get a contradiction. Then, the payoff from the interior solution is increasing in $\pi$ for all $\pi \geq \bar{\pi}(v^h, v^l)$, hence for all such $\pi$, the seller strictly prefers the interior solution.

Now, let me do a similar exercise for $v^h$ (now, I drop $\pi$ and $v^l$ as the arguments of the analyzed functions). Suppose the seller strictly prefers the interior solution at some $v^h$, then:

$$\frac{d\tilde{V}^S}{dv^h}(q_a^o(v^h), v^h) = \frac{\partial \tilde{V}^S}{\partial v^h}(q_a^o(v^h), v^h) = \frac{\pi}{1-\pi}\Big/\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(v^h)}{q_a^o(v^h)}\right) + \frac{1}{1-\pi}\right) - q_a^o(v^h) > 0$$

where the inequality is again obtained by bounding $\ln(y) \leq y - 1$ . Hence, the seller's payoff from an interior solution is increasing in $v^h$.Then, if the seller prefers the interior solution at $\tilde{v}^h$, she strictly prefers an interior solution for all higher $v^h$.

□

*Proof of Proposition 5.* I now derive the comparative statics of $q_a^o$ with respect to $\pi$ and $v^h$.
**Step 1**: comparative statics of $q_a^o$ with respect to $\pi$. I now derive the comparative statics of $q_a^o$ with respect to $\pi$ and $v^h$. When $q_a^o(\pi, v^h, v^l) \neq \pi$ is an optimum, the sign of $\frac{\partial q_a^o}{\partial \pi}$ is determined by the sign of

$$\pi v^h - 2\frac{\frac{\pi}{1-\pi}v^h + v^l}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(v^h)}{q_a^o(v^h)}\right) + \frac{1}{1-\pi}} \underset{\pi\to 1}{\to} v^h - 2v^h < 0$$

where I used Step 1 in Proposition 9, where we verified the seller's payoff at an interior solution converges to $v^h$ as $\pi \to 1$.
**Step 2**: comparative statics of $q_a^o$ with respect to $v^h$. At the interior candidate solution (when it is optimal), the sign of $\frac{\partial q_a^o}{\partial v^h}$ is given by $\frac{\partial^2 \tilde{V}^S}{\partial q^o \partial v^h}$ evaluated at $q_a^o(v^h)$:

$$\frac{\partial^2 \tilde{V}^S}{\partial q^o \partial v^h} = \frac{\pi}{1-\pi} / \left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(v^h)}{q_a^o(v^h)}\right) + \frac{1}{1-\pi}\right)^2 \frac{1}{(1-q_a^o(v^h))q_a^o(v^h)} - 1 =$$
$$\frac{\pi}{1-\pi}(v^h - v^l) / \left(\frac{\pi}{1-\pi}v^h + v^l\right) - 1 < 0$$

**Step 3**: comparative statics of $\mathbf{p}(x)$ in the optimal uniform-threshold market outcome with respect to $v^h$.

With some abuse of notation, let $\mathbf{p} : X \times \mathbb{R} \to \mathbb{R}$, and $\mathbf{q} : [0,1) \times \mathbb{R} \to \mathbb{R}$ denote the price schedule and quality composition for every store location given the value of a high-quality product in an optimal market outcome from by Theorem 2. Similarly, $\hat{x} : \mathbb{R} \to (0,1)$ stands for an optimal earliest outlet location for every given $v^h$. To show that the price increases in every store location, recall that for all non-outlet locations, the price schedule satisfies:

$$\mathbf{p}(x, v^h) = v^h - \frac{q_a^o(v^h)}{\mathbf{q}(x, v^h)}(v^h - v^l)$$
$$\Rightarrow \frac{\partial \mathbf{p}(x, v^h)}{\partial v^h} = 1 - \frac{q_a^o(v^h)}{\mathbf{q}(x, v^h)} - (v^h - v^l)\frac{\partial}{\partial v^h}\left(\frac{q_a^o(v^h)}{\mathbf{q}(x, v^h)}\right), \forall x < \hat{x}(v^h)$$

Since $q_a^o(v^h) \leq \mathbf{q}(x, v^h), \forall x \in X$, to verify $\mathbf{p}(x, v^h)$ increases with $v^h$, it is sufficient to check if $\frac{\partial}{\partial v^h}\left(\frac{\mathbf{q}(x,v^h)}{q_a^o(v^h)}\right) \geq 0$.

Define $r(x, v^h) \equiv \frac{\mathbf{q}(x,v^h)}{q_a^o(v^h)}$. Given the steady-state condition on $\mathbf{q}(\cdot,\cdot)$:

$$\begin{aligned}\frac{\partial^2 r(x,v^h)}{\partial x \partial v^h} &= \frac{\partial}{\partial v^h}\left(-\frac{\mathbf{q}(x,v^h)(1-\mathbf{q}(x,v^h))^2}{q_a^o(v^h)(1-q_a^o(v^h))(1-\hat{x}(v^h))}\right) = \frac{\partial}{\partial v^h}\left(-r(x,v^h)\frac{(1-q_a^o(v^h)r(x,v^h))^2}{(1-q_a^o(v^h))(1-\hat{x}(v^h))}\right) \\ &= -\frac{\left(1-q_a^o(v^h)r(x,v^h)\right)\left[1-2q_a^o(v^h)r(x,v^h)\right]}{(1-q_a^o(v^h))(1-\hat{x}(v^h))}\frac{\partial r(x,v^h)}{\partial v^h} + \frac{2r(x,v^h)\left(1-q_a^o(v^h)r(x,v^h)\right)}{(1-q_a^o(v^h))(1-\hat{x}(v^h))}\frac{\partial q_a^o(v^h)}{\partial v^h} \\ &\quad + r(x,v^h)\frac{(1-q_a^o(v^h)r(x,v^h))^2}{(1-q_a^o(v^h))^2(1-\hat{x}(v^h))^2}\frac{\partial\left((1-q_a^o(v^h))(1-\hat{x}(v^h))\right)}{\partial v^h}\end{aligned} \tag{10}$$

As shown above, $\frac{\partial q_a^o(v^h)}{\partial v^h} < 0$. In addition, $\hat{x}(v^h)$ and $q_a^o(v^h)$ satisfy:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q_a^o(v^h)}{q_a^o(v^h)}\right) + \frac{1}{1-\pi} = \frac{1}{(1-q_a^o(v^h))(1-\hat{x}(v^h))}$$

Hence, if $\frac{\partial q_a^o(v^h)}{\partial v^h} < 0$, then $\frac{\partial\left((1-q_a^o(v^h))(1-\hat{x}(v^h))\right)}{\partial v^h} < 0$ and $\frac{\partial \hat{x}(v^h)}{\partial v^h} > 0$. Since the earliest outlet locations shifts to the right, $r(\hat{x}(v^h), v^h+\varepsilon) > 1 = r(\hat{x}(v^h), v^h)$. That is:

$$\frac{\partial r}{\partial v^h}(\hat{x}(v^h), v^h) > 0$$

Hence, $\frac{\partial r}{\partial v^h}$ has the desired sign at least in the left neighborhood of $\hat{x}(v^h)$. Note that $\frac{\partial r}{\partial v^h}$ can never cross 0, since otherwise there exists $\tilde{x}$, such that $\frac{\partial r}{\partial v^h}(\tilde{x}, v^h) = 0$, and is positive in the right neighborhood of $\tilde{x}$. However, this is not possible by Equation (10), as $\frac{\partial r}{\partial v^h \partial x}(\tilde{x}, v^h) < 0$. This completes the proof. □

# I Omitted proofs for Section 4.1

*Proof of Proposition 6.* Consider a market outcome $\mathbf{p} \in A$, $D, \mathbf{q} \in \mathcal{E}_{\mathbf{p},\gamma}$ with the earliest outlet location $\hat{x}_l$. Let $q^o = \mathbf{q}(\hat{x}_l)$. By Lemma 15, given $\gamma$, the highest measure of outlet shoppers $\bar{D}_o$ that could still induce $q^o$ is given by:

$$\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o}{q^o}\right) + \frac{1}{1-\pi}\right] = \frac{1+\gamma}{(\bar{D}_o+\gamma)(1-q^o)}$$

By Lemma 14, the total surplus at the market outcome with positive sales is given by:

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) = (D_o+\gamma)(1-q^o)\left(\frac{\pi}{1-\pi}v^h + v^l\right) - \gamma q^o(v^h - v^l) - \gamma(\kappa + v^l)$$

Given the bound on $D_o \leq \bar{D}_o$, the above is at most:

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) \leq \left(\frac{\pi}{1-\pi}v^h + v^l\right) \frac{1+\gamma}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o}{q^o}\right) + \frac{1}{1-\pi}} - \gamma q^o(v^h - v^l) - \gamma(\kappa + v^l)$$

The seller's payoff is the difference between the total surplus and the consumer payoff:

$$\begin{aligned} V^S(\mathbf{p}, D, \mathbf{q}, \gamma) \leq \left(\frac{\pi}{1-\pi}v^h + v^l\right) \frac{1+\gamma}{\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o}{q^o}\right) + \frac{1}{1-\pi}} - \gamma q^o(v^h - v^l) - \gamma(\kappa + v^l) - q^o(v^h - v^l) \\ = (1+\gamma)\tilde{V}^S(q^o) - \gamma(\kappa + v^l) \end{aligned}$$

By Lemma 15 again, for every $\gamma$, $q^o \geq Q(\gamma)$, where $Q(\gamma)$ is defined as:

$$\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-Q(\gamma)}{Q(\gamma)}\right) + \frac{1}{1-\pi}\right] = \frac{1+\gamma}{\gamma(1-Q(\gamma))}$$

Hence, the seller's maximal profit among all market outcomes that use outlet locations is at most:

$$V^* = \sup_{\gamma \geq 0} \sup_{\substack{q^o \in [Q(\gamma), \pi] \\ q^o > 0}} (1+\gamma)\tilde{V}^S(q) - \gamma(\kappa + v^l)$$

Alternatively, suppose that consumers only shop at non-outlet locations ($D_o = 0$). In this case, the seller may extract the whole total surplus from a market outcome by charging a price of $v^h$ at all store locations. The seller's optimal profit among such market outcomes is:

$$\begin{aligned} V^{**} &= \sup_{\gamma > 0} \hat{V}^S(\gamma, \kappa) \\ \text{where } \hat{V}^S(\gamma, \kappa) &= \left(\frac{\pi}{1-\pi}v^h \gamma(1 - Q(\gamma)) - Q(\gamma)v_h\gamma - \gamma\kappa\right) \end{aligned}$$

Finally, whenever consumers shop only at outlet locations, the seller receives a constant price $\bar{p} \leq v^l$, and the seller's profit is at most $v^l \leq V^* = \tilde{V}^S(\pi)$.

Let $q^*$ be the optimizer of $\tilde{V}^S$ over $(0, \pi]$. I now establish the following:

$$\max\{V^*, V^{**}\} = \begin{cases} \tilde{V}^S(q^*), \text{ if } \pi/(1-\pi)v^h \leq \kappa \\ \max_{\gamma \in (0,\infty)} \hat{V}^S(\gamma, \kappa), \text{ if } \tilde{V}^S(q^*) > \kappa + v^l \\ \max\left\{\tilde{V}^S(q^*), \max_{\gamma \in (0,\infty)} \hat{V}^S(\gamma, \kappa)\right\}, \text{ if } \kappa \in \left[\tilde{V}^S(q^*) - v^l, \frac{\pi}{1-\pi}v^h\right) \end{cases}$$

**Case 1**. Note that whenever $\tilde{V}^S(q^*) > \kappa + v^l$, the seller is better off not having outlet locations: $V^{**} > V^*$. Indeed, note first that $Q(\gamma)$ is increasing $\gamma$. In addition, $\lim_{\gamma\to 0} Q(\gamma) = 0$ and $\lim_{\gamma\to\infty} Q(\gamma) = \pi$. Hence, there exists a unique $\tilde{\gamma} \in (0,\infty)$, such that $Q(\tilde{\gamma}) = q^*$. Then, for all $\gamma \leq \tilde{\gamma}$:

$$(1+\gamma)\tilde{V}^S(q^o) - \gamma(\kappa+v^l) \leq (1+\gamma)\tilde{V}^S(q^*) - \gamma(\kappa+v^l) \leq (1+\tilde{\gamma})\tilde{V}^S(Q(\tilde{\gamma})) - \tilde{\gamma}(\kappa+v^l)$$
$$< \left(\frac{\pi}{1-\pi}v^h\tilde{\gamma}(1-Q(\tilde{\gamma})) - Q(\tilde{\gamma})v_h\tilde{\gamma} - \tilde{\gamma}\kappa\right) \leq V^{**}$$

Similarly, for all $\gamma > \tilde{\gamma}$,

$$\sup_{\substack{q^o\in[Q(\gamma),\pi]\\ q^o>0}} (1+\gamma)\tilde{V}^S(q^o) - \gamma(\kappa+v^l) = (1+\gamma)\max\{\tilde{V}^S(\pi), \tilde{V}^S(Q(\gamma))\} - \gamma(\kappa+v^l) < V^{**}$$

where I use the fact that $\tilde{V}^S$ is convex-concave by Lemma 16, and hence whenever $Q(\gamma)$ is binding, $\tilde{V}^S$ reaches its optimum at one of the corners. That is, $\tilde{V}^S(q^*) > \kappa + v^l$ is sufficient for the seller not to use outlet locations.

**Case 2**. Alternatively, suppose $\tilde{V}^S(q^*) \leq \kappa + v^l$, then:

$$\sup_{\substack{q^o\in[Q(\gamma),\pi]\\ q^o>0}} (1+\gamma)\tilde{V}^S(q^o) - \gamma(\kappa+v^l) \leq (1+\gamma)\tilde{V}^S(q^*) - \gamma(\kappa+v^l) \leq \tilde{V}^S(q^*)$$

That is, in this case, $V^* = \tilde{V}^S(q^*)$, and the seller does not use direct disposal simultaneously with outlet locations. Analogously, we can verify that for $\pi/(1-\pi)v^h \leq \kappa$, the seller prefers to have a positive measure of shoppers at outlet locations, since $V^* \geq \tilde{V}^S(\pi) = v^l > 0 = V^{**}$.

**Optimal Disposal Rate**. Let me now verify that an optimal $\hat{V}^S$ attains its optimum on $(0,\infty)$ for any $\kappa > 0$ whenever $\pi/(1-\pi)v^h > \kappa$. Consider:

$$\frac{\partial\hat{V}^S}{\partial\gamma} = \frac{\pi}{1-\pi}v^h(1-Q(\gamma)) - Q(\gamma)v_h - \kappa - \frac{\gamma v^h}{1-\pi}Q'(\gamma)$$
$$\underset{\gamma\to\infty}{\to} -\kappa - \lim_{\gamma\to\infty}\frac{\gamma v^h}{1-\pi}Q'(\gamma) \leq -\kappa$$
$$\underset{\gamma\to 0}{\to} \frac{\pi}{1-\pi}v^h - \kappa - \lim_{\gamma\to 0}\frac{\gamma v^h}{1-\pi}Q'(\gamma) > 0$$

Hence, the unbounded disposal rate is suboptimal. To establish the optimal choice of $\gamma$ is strictly above 0 for any $\pi/(1-\pi)v^h > \kappa$, it is enough to show $\lim_{\gamma\to 0}\gamma Q'(\gamma) = 0$.

$$Q'(\gamma)\gamma = \frac{1}{(1+\gamma)/(1-Q(\gamma)) + \gamma/Q(\gamma)}$$

Hence, the limit $\lim_{\gamma\to 0}\gamma Q'(\gamma) = 0$ is determined by $\gamma/Q(\gamma)$. From the definition of $Q$, it must be that $\gamma$ converges to 0 at the same rate as $\ln(Q(\gamma))$, hence $\lim_{\gamma\to 0}\frac{\gamma}{Q(\gamma)} = \infty$ implying $\lim_{\gamma\to 0}\gamma Q'(\gamma) = 0$ as required. That is, $\hat{V}^S(\gamma,\kappa)$ attains its maximum for every $\kappa \in \left(0, \frac{\pi}{1-\pi}v^h\right)$.

**Solution Switches**. Finally, to establish there is a unique threshold where the optimal solution switches, note that $\tilde{V}^S(q^*)$ is independent of $\kappa$ whereas $\max_{\gamma\in(0,\infty)}\hat{V}^S(\gamma,\kappa)$ is strictly decreasing in $\kappa$. Hence, there exists $\bar{\kappa}(v^h, v^l, \pi)$ as in the formulation of the proposition.

To prove $\bar{\kappa}$ is increasing in $v^l$, note that $\max_{\gamma\in(0,\infty)}\hat{V}^S(\gamma,\kappa)$ is constant in $v^l$, but $\tilde{V}^S(q^*)$ is strictly increasing in $v^l$. Hence, if $v^l$ increases, the seller is indifferent between the two regimes at a higher production/disposal cost.

□

## J Omitted Proofs for Section 4.2

*Proof of Theorem 3.* If the sales are zero, then all locations are non-outlet locations and the outlet threshold is $\hat{x} = 1$. Otherwise, if the total sales are positive, then the market outcome is $\hat{x}$-threshold market outcome for some $\hat{x}$ due to Lemma 13.

The total surplus satisfies the formulation of the theorem due to Lemma 14.

Let $\int_{\mathbf{p}(y)\le v^l} dD(y) = D_o$.

**Case 1.**: $D_o > 0$, then consumers shop at outlet locations with a positive probability (for both market outcomes). By Lemma 13 (ii) and Lemma 11 (ii), all these locations have quality $\mathbf{q}(\hat{x})$ ($D$-a.s.).

**Case 1.1**: $D_o < 1$. In this case, by Lemma 13 (iii), consumers shop at prices (weakly) above $v^l$ with probability 1, under both market outcomes. Hence, the consumers' payoffs are given by $V^B(\mathbf{p}, D, \mathbf{q}, \gamma) = \mathbf{q}(\hat{x})(v^h - v^l)$. Consequently, the seller's payoff equals:

$$V^S(\mathbf{p}, D, \mathbf{q}, \gamma) = TS(\mathbf{p}, D, \mathbf{q}, \gamma) - \mathbf{q}(\hat{x})(v^h - v^l)$$

I now show the additional part of the theorem. First, any market outcome with zero sales is suboptimal, so let me restrict attention to the outcomes with positive sales. $D$ can only have discontinuities at non-outlet locations if a positive mass of consumers shop there. If $D_o > 0$, from the above analysis, the seller's payoff is

$$V^S(\mathbf{p}, D, \mathbf{q}, \gamma) = (D_o + \gamma)(1 - \mathbf{q}(\hat{x}))\left(\frac{\pi}{1-\pi}v^h + v^l\right) - (1+\gamma)\mathbf{q}(\hat{x}(v^h - v^l) - \gamma(\kappa + v^l)$$

which is decreasing in $\mathbf{q}(\hat{x}$. From Lemma 15, $\mathbf{q}(\hat{x} < \tilde{q}^o$ for $\tilde{q}^o$:

$$1 + \gamma = (D_o + \gamma)(1 - \tilde{q}^o)\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-\tilde{q}^o}{\tilde{q}^o}\right) + \frac{1}{1-\pi}\right]$$

Similar to Proposition 4, we can construct a market outcome with a uniform shopping strategy that will result in the quality composition $\tilde{q}^o$.

If $D_o = 0$, then the seller may extract the whole surplus with a constant price $v^h$:

$$V^S(\mathbf{p}, D, \mathbf{q}, \gamma) \leq TS(\mathbf{p}, D, \mathbf{q}, \gamma) = \gamma(1 - \mathbf{q}(\hat{x}))\left(\frac{\pi}{1-\pi}v^h + v^l\right) - \gamma\mathbf{q}(\hat{x})(v^h - v^l) - \gamma(\kappa + v^l)$$

As $TS$ is decreasing in $\mathbf{q}(\hat{x})$, the seller again would benefit by deviating to $\tilde{\mathbf{p}}(x) = v^h, \tilde{\sigma} = 1$ with the same disposal rate $\gamma$.

□

## K Omitted Proofs for Section 4.3

Consider a market outcome $\mathbf{p} \in \mathcal{A}, (\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p},\gamma}$ and the partition of the locations as in Lemma 4: $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \leq v^i\}$, with a convention that $\hat{x}_i = 1$ whenever $\mathbf{p}(x) > v^i$ for all $x \in X$, and $x_i = 0$.

Define the lowest-always-purchased (LAP) $v^j$ such that a product $v^j$ is purchased with probability one when found:

$$j \equiv \max_{n \geq i \geq 1}\left\{i : \int_0^{\hat{x}_i} \sigma(y)dy = 0\right\}$$

And define the lowest-ever-purchased (LEP) product-value $v^J$ as the lowest quality that can ever be purchased with positive probability given the price schedule $\mathbf{p}$:

$$J \equiv \max_{n \geq i \geq 1}\{i : \hat{x}_i < 1\}$$

**Lemma 17.** *Consider a market outcome $\mathbf{p} \in \mathcal{A}, (\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p},\gamma}$ with consumer surplus $CS \geq 0$. Then, for any such market outcome,*

*i) Then, LAP $j$ satisfies:*

$$\sum_{k \leq j} \pi(k)(v^k - v^{j+1}) \geq CS \geq \sum_{k \leq j} \pi(k)(v^k - v^j)$$

*ii) Define for all $i \geq j + 1$:*

$$\rho_{mi} = \frac{CS}{E_{i-1} - v^i}$$

$$E_i = \frac{\rho_m^i E_{i-1} + (1-\rho_m^i)\frac{\pi(i)}{\sum_{l\leq i}\pi(l)} v^l}{\rho_m^i + (1-\rho_m^i)\frac{\pi(i)}{\sum_{l\leq i}\pi(l)}}$$

$$\textit{with } E_j = \frac{\sum_{l\leq j}\pi(k)(v^k - v^j)}{\sum_{l\leq j}\pi(k)}$$

*then for all $i$ such that $\hat{x}_i < 1$, we must have:*

$$\rho_m(\hat{x}_i -) = \rho_m^i$$

$$\rho_m(\hat{x}_i) = \rho_m^i + (1-\rho_m^i)\frac{\pi(i)}{\sum_{j\leq i}\pi(j)}$$

*iii) Let $\rho_j = \sum_{l<j}\pi(l)$, then LEP $J$ is such that for all $k \leq (>) J$:*

$$\frac{1-\rho_k}{\sum_{m\geq k}\pi(m)}\Bigg[\sum_{i=j+1}^{k}\Bigg(\ln\Bigg(\Bigg(\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}} + \frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)}\Bigg)\frac{1-\rho_m^i}{\rho_m^i}\Bigg) + \frac{\rho_m^{i-1}}{1-\rho_m^{i-1}} + \frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)} - \frac{\rho_m^i}{1-\rho_m^i}\Bigg)\sum_{l\geq i}\pi(l)\Bigg] \leq (>) \frac{1}{\gamma}$$

*In addition, if $J = n$, then the share of consumers shopping at a price of at most $v^n$ is:*

$$\frac{1-\rho_n}{\sum_{m\geq n}\pi(m)}\Bigg[\sum_{i=j+1}^{n}\Bigg(\ln\Bigg(\Bigg(\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}} + \frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)}\Bigg)\frac{1-\rho_m^i}{\rho_m^i}\Bigg) + \frac{\rho_m^{i-1}}{1-\rho_m^{i-1}} + \frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)} - \frac{\rho_m^i}{1-\rho_m^i}\Bigg)\sum_{l\geq i}\pi(l)\Bigg] = \frac{1-D_o}{\gamma + D_o}$$

*Proof.* (i) Let $\bar{y}_j = \sup\{\int_0^{\bar{y}_j}\sigma(y)dy = 0\}$. Then, it must be that in any right neighborhood of $\bar{y}_j$, the consumer shops with positive probability, and we can find a sequence $\{z_k\}$ with $z_k \to \bar{y}_j$, such that the consumer's payoff is given by :

$$CS = \sum_{i=1}^{n}\mathbf{q}(i|z_k)(v^i - \mathbf{p}(z_k))_+$$

In addition, by Lemma 4 $\hat{x}_i$, $\mathbf{p}(x) \in (v^{j+1}, v^j)$ $D$-a.e. $[\hat{x}_j, \hat{j}_{l+1}]$. Hence, the consumer payoff along the sequence is bounded by:

$$\sum_{i=1}^{n} \mathbf{q}(i|z_k)(v^i - v^j)_+ \leq CS \leq \sum_{i=1}^{n} \mathbf{q}(i|z_k)(v^i - v^{j+1}))_+$$

As $z_k \to \bar{y}_j$, $\mathbf{q}(i|z_k) \to \mathbf{q}(i|\bar{y}_j)$ by continuity of $\mathbf{q}(i|\cdot)$ due to Lemma 10. In addition, by Lemma 11, $\mathbf{q}(i|\bar{y}_j) = \mathbf{q}(i|0) = \pi(i)$, since $\mathbf{q}$ remains constant over $[0, \bar{y}_j]$. Hence, we obtain:

$$\sum_{k \leq j} \pi(k)(v^k - v^j) \leq CS \leq \sum_{k \leq j} \pi(k)(v^k - v^{j+1})$$

as required.

(ii) **Step 1**. At a boundary $\hat{x}_i$, the probability of purchase jumps upwards by $\mathbf{q}(i|\hat{x})$, as consumers start purchasing a lower quality. Given Lemma 11 (ii), there is no learning about quality $i$ relative to any lower quality on $(0, \hat{x}_i)$, hence we must have:[18]

$$\frac{\mathbf{q}(v^i|\hat{x}_i)}{\pi(i)} = \frac{(1 - \rho(\hat{x}^i-))}{\sum_{l \leq j} \pi(l)}$$

Then, we get the following boundary conditions on the purchasing probability:

$$\rho(\hat{x}_i) = \rho(\hat{x}_i-) + \mathbf{q}(i|\hat{x}_i) = \rho(\hat{x}_i-) + (1 - \rho(\hat{x}^i-))\frac{\pi(i)}{\sum_{l \leq i} \pi(l)}$$

In addition, by Lemma 11 (ii), the expected value of the product of a product conditional on purchase is constant on $[\hat{x}_i, \hat{x}_{i+1})$ and must satisfy: conditions:

$$E_i = \frac{\rho(\hat{x}_i-)E_{i-1} + (1 - \rho(\hat{x}_i-))\frac{\pi(i)}{\sum_{l \leq i} \pi(l)} v^l}{\rho(\hat{x}_i-) + (1 - \rho(\hat{x}_i-))\frac{\pi(i)}{\sum_{l \leq i} \pi(l)}}$$

Again, we use the fact that the consumer starts purchasing a quality $i$ at the boundary $\hat{x}_i$.

**Step 2**. Now, I will show that for every $i$ such that $\hat{x}_i < 1$, we must have

$$CS = \rho(\hat{x}_i-)(E_{i-1} - v^i)$$
$$\int_{\hat{x}_{i-1}}^{\hat{x}_i} \sigma(y)dy > 0$$

I establish the above by induction.

[18] Where we use $\mathbf{q}(l|\cdot)$ is continuous on $(0, \hat{x}_n)$ by Observations 1 and 2.

**Initial Iteration** $i = j+1$. First, let me consider $\hat{x}_{j+1}$. Either $\hat{x}_{j+1} = 1$, so that the consumer only purchases the products whose value is at least $v^j$, or else given quality composition is continuous, the consumer can attain a payoff arbitrarily close to:

$$\rho(\hat{x}_{j+1}-)(E_j - v^{j+1}) \leq V^B(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$$

Let $y = \sup_{x \in [0, \hat{x}^{j+1}]} \left\{ \int_{z \in (y, \hat{x}^{j+1}]} \sigma(y) dy = 0 \right\}$. Then, consumers do not shop over $[y, \hat{x}^{j+1}]$ with probability one, and $\rho(y) = \rho(\hat{x}_{j+1}-)$. In addition, it must be that consumers shop with a positive probability in the right neighborhood of $y$, hence as the consumer's shopping strategy os optimal, there is a sequence $\{z_l\}$ converging to $y$, such that

$$CS = \rho(z_l)(E_j - \mathbf{p}(z_l)) \leq \rho(z_l)(E_j - v^{j+1}) \underset{z_l \to y}{\to} \rho(\hat{x}_{j+1}-)(E_j - v^{j+1})$$

Hence, given $V^B(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$, the purchasing probability at $\hat{x}^{j+1}$ satisfies

$$\rho(\hat{x}_{j+1}-)(E_j - v^{j+1}) = CS$$

**Iteration** $i$. Suppose the statement is true for all $k \leq i-1$ for some $i-1 \geq j+1$. If $\hat{x}_i = 1$, the statement is trivially true. Otherwise, suppose that $\hat{x}_i < 1$, then the consumer can get a payoff arbitrarily close to

$$\rho(\hat{x}_i-)(E_{i-1} - v^i) \leq V^B(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$$

If $\int_{\hat{x}_{i-1}}^{\hat{x}_i} \sigma(y) dy = 0$, the quality composition remains the same over an interval $[\hat{x}_{i-1}, \hat{x}_i]$ due to Lemma 11. In this case, due to Step 1, we get:

$$\rho(\hat{x}_i-)(E_{i-1} - v^i) = \rho(\hat{x}_{i-1}+)(E_{i-1} - v^i) = \rho(\hat{x}_{i-1}-)(E_{i-2} - v^i)$$

which implies

$$CS = \rho(\hat{x}_{i-1}-)(E_{i-2} - v^{i-1}) < \rho(\hat{x}_{i-1}-)(E_{i-2} - v^i) = \rho(\hat{x}_i-)(E_{i-1} - v^i)$$

where the first equality is due to the initial hypothesis of our proof by induction. Hence, we may conclude that $\int_{\hat{x}_{i-1}}^{\hat{x}_i} \sigma(y) dy > 0$. This completes the proof by induction.

Combining steps 1 and 2, part (ii) of the Lemma follows.

(iii) By Lemma 11, $(1-\rho(x))S_m(x)$ remains constant over any $(\hat{x}_{i-1}, \hat{x}_i)$, in addition since

$S'_m(x) = -\rho(x)\sigma(x)$, we obtain that over $(\hat{x}_{i-1}, \hat{x}_i)$:

$$\rho'(x) = -\rho(x)(1-\rho(x))\frac{\sigma(x)}{S_m(x)}$$

Recall that $J$ is the lowest quality that is purchased with positive probability, then we have $\sum_{l\geq J} \mathbf{q}(l|x)S_m(x)$ remains constant over $[0, \hat{x}_j)$. In addition, by Lemma 11, for $x \in (\hat{x}_{i-1}, \hat{x}_i)$:

$$(1-\rho(x)) = \sum_{l\leq J} \mathbf{q}(l|x)\frac{\sum_{l\geq i}\pi(l)}{\sum_{m\geq J}\pi(m)}$$

so that we obtain

$$\rho'(x) = -\rho(x)(1-\rho(x))^2\frac{\sigma(x)}{(1-\rho(\hat{x}_J-))S_m(\hat{x}_J)}\frac{\sum_{m\geq J}\pi(m)}{\sum_{l\geq i}\pi(l)}$$

That is, in a market outcome $\mathbf{p} \in \mathcal{A}, (\sigma, \mathbf{q}) \in \mathcal{E}_{\mathbf{p},\gamma}$, we must have:

$$\frac{\sum_{l\geq i}\pi(l)}{\sum_{m\geq J}\pi(m)}\left(\ln\left(\frac{\rho(\hat{x}_{i-1})}{1-\rho(\hat{x}_{i-1})}\frac{1-\rho(\hat{x}_i-)}{\rho(\hat{x}_i-)}\right)+\frac{\rho(\hat{x}_{i-1})}{1-\rho(\hat{x}_{i-1})}-\frac{\rho(\hat{x}_i-)}{1-\rho(\hat{x}_i-)}\right) = \frac{D(\hat{x}_i)-D(\hat{x}_{i-1})}{(1-\rho(\hat{x}_J-))S_m(\hat{x}_J)}$$

As $D(\hat{x}_J) \leq 1$ and $S_m(\hat{x}_J) \geq \gamma$, then from the above:

$$(1-\rho(\hat{x}_J-))\left[\sum_{i=j+1}^{J}\frac{\sum_{l\geq i}\pi(k)}{\sum_{m\geq J}\pi(l)}\left(\ln\left(\frac{\rho(\hat{x}_{i-1})}{1-\rho(\hat{x}_{i-1})}\frac{1-\rho(\hat{x}_i-)}{\rho(\hat{x}_i-)}\right)+\frac{\rho(\hat{x}_{i-1})}{1-\rho(\hat{x}_{i-1})}-\frac{\rho(\hat{x}_i-)}{1-\rho(\hat{x}_i-)}\right)\right] = \frac{D(\hat{x}_J)}{S_m(\hat{x}_J)} \leq \frac{1}{\gamma}$$

Combining with part (ii), we obtain that for $J$, the desired inequality must hold:

$$\frac{1-\rho_J}{\sum_{m\geq J}\pi(m)}\left[\sum_{i=1}^{J}\left(\ln\left(\left(\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}+\frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)}\right)\frac{1-\rho_m^i}{\rho_m^i}\right)+\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}} + \frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)}-\frac{\rho_m^i}{1-\rho_m^i}\right)\sum_{l\geq i}\pi(l)\right] \leq \frac{1}{\gamma}$$

As $\rho_m^i$ are $\sum_{k\geq i}\pi(k)$ are both decreasing $i$, then for all $k \leq J$:

$$\frac{1-\rho_k}{\sum_{m\geq k}\pi(m)}\left[\sum_{i=1}^{k}\left(\ln\left(\left(\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}+\frac{\pi(i-1)}{\sum_{j\leq i-1}\pi(j)}\right)\frac{1-\rho_m^i}{\rho_m^i}\right)+\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}\right.\right.$$

$$+\frac{\pi(i-1)}{\sum_{j\leq i-1}\pi(j)}-\frac{\rho_m^i}{1-\rho_m^i}\Bigg)\sum_{l\geq i}\pi(l)\Bigg]\leq\frac{1}{\gamma}$$

If $J=n$, then we are done, and the statement is true. In addition, $S_m(\hat{x}_J)=D_o+\gamma$, and we obtain:

$$\frac{1-\rho_J}{\sum_{m\geq J}\pi(m)}\Bigg[\sum_{i=1}^{J}\Bigg(\ln\Bigg(\Bigg(\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}+\frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)}\Bigg)\frac{1-\rho_m^i}{\rho_m^i}\Bigg)+\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}$$
$$+\frac{\pi(i-1)}{\sum_{l\leq i-1}\pi(l)}-\frac{\rho_m^i}{1-\rho_m^i}\Bigg)\sum_{l\geq i}\pi(l)\Bigg]=\frac{1-D_o}{\gamma+D_o}$$

Otherwise, let $\bar{x}=\inf\{x\in X: D(x)=1\}$. In this case, $S_m(\bar{x})=\gamma$ and $D(\bar{x})=1$, then as we obtain that:[19]

$$\frac{1-\rho(\bar{x})}{\sum_{m\geq J+1}\pi(m)}\Bigg[\Bigg(\ln\Bigg(\Bigg(\frac{\rho_m^{J}}{1-\rho_m^{J}}+\frac{\pi(J)}{\sum_{l\leq J}\pi(l)}\Bigg)\frac{1-\rho(\bar{x})}{\rho(\bar{x})}\Bigg)+\frac{\rho_m^{J}}{1-\rho_m^{J}}$$
$$+\frac{\pi(J)}{\sum_{l\leq J}\pi(l)}-\frac{\rho(\bar{x})}{1-\rho(\bar{x})}\Bigg)\frac{1}{\sum_{l\geq J+1}\pi(l)}$$
$$+\sum_{i=1}^{J}\Bigg(\ln\Bigg(\Bigg(\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}+\frac{\pi(i-1)}{\sum_{j\leq i-1}\pi(j)}\Bigg)\frac{1-\rho_m^i}{\rho_m^i}\Bigg)+\frac{\rho_m^{i-1}}{1-\rho_m^{i-1}}$$
$$+\frac{\pi(i-1)}{\sum_{j\leq i-1}\pi(j)}-\frac{\rho_m^i}{1-\rho_m^i}\Bigg)\frac{1}{\sum_{l\geq i}\pi(l)}\Bigg]=\frac{1}{\gamma}$$

Then as the price never takes values below $v^{J+1}$ on $X$. Hence, it must be that $\rho_m(\bar{x})>\rho_m^k$ for every $k\geq J+1$ and the result follows.

□

*Proof of Proposition 7.* By Lemma 13, for any market outcome $\mathbf{p}\in\mathcal{A},(\sigma,\mathbf{q})\in\mathcal{E}_{\mathbf{p},\gamma}$ there exists $j$ and $J$ such that only products with $i\leq J$ are purchased with positive probability (and the existence of a threshold price follows); all products $i\leq j$ are purchased with probability one when found.

By Lemma 3*, either the market outcome has zero total steady-state sales, or $S_m(\hat{x}_n)>0$. Whenever $\gamma=0$, $S_m(\hat{x}_n)>0$ only when there is a non-trivial mass of consumers shopping at a price of at most $v^n$.

[19] We now use that $\sum_{l\geq J+1}\mathbf{q}(l|x)S_m(x)$ remains constant over $[0,\bar{x})$.

Moreover, Lemma 17, for every $CS$ such $j$ and $J$ are uniquely defined.

By Lemma 11, the expected value conditional on purchase, is constant on $(\hat{x}_i, \hat{x}_{i+1})$. Applying Lemma 17, we can express the total surplus as follows:

$$\begin{aligned}
TS(\mathbf{p}, \sigma, \mathbf{q}, \gamma) &= \sum_{i=j}^{J} \int_{\hat{x}_{i-1}}^{\hat{x}_i} \rho_m(x) E_i \sigma(x) dx + D_o E_n \\
&= \sum_{i=j+1}^{J} E_i \left(S(\hat{x}_{i-1}) - S_m(\hat{x}_i)\right) + D_o E_n \\
&= (D_o + \gamma)(1 - \rho_m^J) \sum_{i=j+1}^{J} E_i \left( \frac{\rho_m^i}{1 - \rho_m^i} - \frac{\rho_m^{i-1}}{1 - \rho_m^{i-1}} - \frac{\pi(i-1)}{\sum_{l \leq i-1} \pi(l)} \right) + D_o E_n
\end{aligned}$$

where $\rho_m^i$ and $E_i$ are uniquely defined for a given consumer surplus $CS$. Then, the total consumer surplus only depends on the induced consumer surplus $CS$. It follows straightaway that the seller's payoff is the same for any two market outcomes inducing the same consumer surplus $CS$.

□

## L Omitted Proofs for Section 4.4

To use the results from the previous sections, note that we can derive the mass of effective shoppers to be:

$$D(x) = \int_{\theta : \mathbf{x}(\theta) \leq x} \mathbb{1}\{\mathbf{p}(\mathbf{x}(\theta)) \leq \theta\} f(\theta) d\theta$$

Neither of the results in Appendix F rely on $D(1) = 1$ (rather than any smaller); hence they readily apply to $D(\cdot)$ as specified above.

In particular, from Lemma 3* in any market outcome with non-zero sales, there is a positive mass of consumers shopping at outlet locations. If $\hat{x} = \inf\{x : \mathbf{p}(x) \leq v^l\}$, then $\mathbf{q}(\hat{x}) > 0$, then from Lemma 12, $\mathbf{q}(\hat{x}) \equiv q^o > 0$. And from Lemma 13, all locations in $(\hat{x}, 1)$ are outlet locations $D$-a.s.; in addition, almost all such locations hold quality composition $q^o$ by Lemma 11.

**Lemma 18.** *In every market outcome,*

*i) $Q$ is increasing*

*ii) For every* $\theta > v^l$*:*

$$U(\theta) = U(\bar{\theta}) + \int_{\bar{\theta}}^{\theta} Q(s)ds$$

*where*

$$\bar{\theta} = \sup\{\theta : \mathbf{p}(\mathbf{x}(\theta)) \leq v^l\}$$

*Proof.* As $q^o > 0$, and there is a positive measure of outlet locations in $(\hat{x}, 1)$, then any type $\theta > v^l$ receives a strictly positive payoff in every market outcome. Hence, for almost every $\theta$ (apart possible at the boundary where $\theta = v^l$), $\mathbf{p}(\mathbf{x}(\theta)) < \theta$.

Then, from (IC), for any $\theta, \theta' > v^l$, $\theta$ does not have a profitable deviation towards $\mathbf{x}(\theta')$ if and only if:

$$U(\theta) \geq U(\theta') + Q(\theta')(\theta - \theta')$$

Note that for any market outcome with positive total sales, $\bar{\theta} > v^l$, so that by the standard argument we obtain that $Q$ agrees with IC only if $Q$ is increasing and

$$U(\theta) = U(\bar{\theta}) + \int_{\bar{\theta}}^{\theta} Q(s)ds$$

□

**Lemma 19.** *In every market outcome*

*i) if* $\theta < \bar{\theta}$*, then* $\mathbf{x}(\theta) \geq \hat{x}$

*ii)* $\mathbf{x}$ *is decreasing on* $(\bar{\theta}, v^h]$

*Proof.* i) Suppose not, and there exists $\tilde{\theta} < \bar{\theta}$, such that $\mathbf{x}(\theta) < \hat{x}$. By definition of $\hat{x}$, then $\mathbf{p}(\mathbf{x}(\tilde{\theta})) > v^l$.

In addition, there exists $\theta'$ arbitrarily close to $\bar{\theta}$ shopping at outlet locations with a quality composition $q^o$. As $\mathbf{p}(\mathbf{x}(\tilde{\theta})) > v^l$, then $\tilde{\theta}$ does not have a profitable deviation to one of the outlet locations only if $Q(\tilde{\theta}) > q^o$. But then we obtain a contradiction with monotonicity of $Q$ from Lemma 18.

ii) By definition of $\bar{\theta}$, all consumer types above $\bar{\theta}$, shop at non-outlet locations in $(0, \hat{x})$. Suppose by a way of contradiction that there exist $\theta_1 > \theta_2 > \hat{\theta}$, such that $\hat{x} > \mathbf{x}(\theta_1) > \mathbf{x}(\theta_2)$. By Lemma 11, $(1 - \mathbf{q}(\mathbf{x}(\theta_1)))S_m(\mathbf{x}(\theta_1)) = (1 - \mathbf{q}(\mathbf{x}(\theta_2)))S_m(\mathbf{x}(\theta_2))$. Hence, to satisfy the monotonicity condition for $Q$, it must be that a zero mass of consumers shop at location in $[\mathbf{x}(\theta_2), \mathbf{x}(\theta_2)]$. This is only possible if a non-trivial subset of consumer types in $(\theta_2, \theta_1)$ shops either at locations $(0, \mathbf{x}(\theta_1))$; or at location $(\mathbf{x}(\theta_2), 1)$.

Either way, we get that there exist some two types $\theta_1' > \theta_2'$, for whom $\mathbf{x}(\theta_1') > \mathbf{x}(\theta_2')$ and there is a non-trivial mass of consumers shopping at $(\mathbf{x}(\theta_2'), \mathbf{x}(\theta_1'))$, which implies $Q(\theta_1') < Q(\theta_2')$ violating monotonicity of $Q$.

□

*Proof of Proposition 8.* (i) Follows from Lemma 19.

(ii) Consider $\theta < \bar{\theta}$. By Lemma 19, $\mathbf{x}$ is decreasing on $(\bar{\theta}, v^h]$. As $\mathbf{x}$ is injective, it is strictly decreasing. Then, a mass consumers shopping between $(\mathbf{x}(\theta), \mathbf{x}(\theta - \Delta)]$ is $\Delta$. By Lemma 11, $(1 - Q(\theta))S_m(\mathbf{x}(\theta)) = (1 - Q(\theta - \Delta))S_m(\mathbf{x}(\theta - \Delta))$.

Then, $\Delta \to 0$, $S_m(\mathbf{x}(\theta - \Delta)) \to S_m(\mathbf{x}(\theta))$, as a vanishing mass of consumers shops between on $(\mathbf{x}(\theta), \mathbf{x}(\theta - \Delta)]$. Hence, $Q$ is right-continuous on $(\bar{\theta}, v^h]$. We can similarly show that for any $\theta \in (\bar{\theta}, v^h)$, $Q$ is also left-continuous.

In addition, we have:

$$Q(\theta) - Q(\theta - \Delta) = (1 - Q(\theta - \Delta))\frac{S_m(\mathbf{x}(\theta)) - S_m(\mathbf{x}(\theta - \Delta))}{S_m(\mathbf{x}(\theta))}$$

As we have:

$$S_m(\mathbf{x}(\theta)) - S_m(\mathbf{x}(\theta - \Delta)) = \int_{\theta-\Delta}^{\theta} f(s)Q(s)ds$$

Then, $Q$ is differentiable a.e. on $(\bar{\theta}, v^h]$ with

$$\begin{aligned} Q'(\theta) &= (1 - Q(\theta))Q(\theta)\frac{f(\theta)}{S_m(\mathbf{x}(\theta))} \\ &= (1 - Q(\theta))^2 Q(\theta)\frac{f(\theta)}{F\left(\bar{\theta}\right) Q\left(\bar{\theta}+\right)} \end{aligned}$$

where we used again that the mass of low-quality products shipped across locations in $(0, \hat{x})$ remains constant, and $S_m(\mathbf{x}(\bar{\theta}+)) = F(\bar{\theta})$, as all type below $\bar{\theta}$ shop at outlet locations, where both types of products are purchased. To establish that $Q$ as in the formulation of the proposition, it remains to establish $Q$ is right-continuous at $\bar{\theta}$. By monotonicity of $Q$, $Q(\bar{\theta}) \leq Q(\bar{\theta}+)$. In addition, $\mathbf{q}(\hat{x}) = Q(\bar{\theta}+)$ by continuity of $\mathbf{q}$ at $\hat{x}$ due to Lemma 10, Lemma 11. Since almost all consumers shopping at outlet locations get quality composition

$\mathbf{q}(\hat{x})$, then due to monotonicity of $Q$, we must require:

$$Q(\bar{\theta}) \geq \mathbf{q}(\hat{x}) = Q(\bar{\theta}+)$$

Hence, $Q(\bar{\theta}) = Q(\bar{\theta}+)$.

(iii) Follows from (ii) and Lemma 18.

(iv) Note that as outlet locations charge a price of at most $v^l$ and almost all of them hold quality composition $Q(\bar{\theta})$, it must be that $U(\bar{\theta}-) \geq Q(\bar{\theta})(\bar{\theta} - v^l)$, whereas if $\bar{\theta} < v^h$, $U(\bar{\theta}+) \leq Q(\bar{\theta})(\bar{\theta} - v^l)$, as all consumer types above $\bar{\theta}$ shop at prices above $v^l$. Hence, whenever $\bar{\theta} < v^h$, we must have:

$$Q(\bar{\theta})(\bar{\theta} - v^l) \geq U(\bar{\theta}+) \geq U(\bar{\theta}-) \geq Q(\bar{\theta})(\bar{\theta} - v^l)$$

which can only hold if $U(\bar{\theta}-) = Q(\bar{\theta})(\bar{\theta} - v^l)$, meaning almost all outlet locations have a price $v^l$.

To prove there exists a market outcome with positive sales for every $\bar{\theta} \in (v^l, v^h]$, let me construct a particular market outcome. Take any such $\bar{\theta}$, and let:

$$\mathbf{x}(\theta) = \frac{1}{3}\left(1 + \frac{\theta - v^h}{v^h - v^l}\right)$$

Let the steady-state quality composition be specified as:

$$\mathbf{q}(x) = \begin{cases} \pi, \text{ if } x \leq 1/3 \\ Q^{\bar{\theta}}(\theta), \text{ if } x \in [1/3, \mathbf{x}\left(\bar{\theta}\right)] \\ Q^{\bar{\theta}}\left(\bar{\theta}\right), \text{ if } x \geq \mathbf{x}\left(\bar{\theta}\right) \end{cases}$$

□