Markdowns

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Abstract

I model markdown pricing as a tool for price discrimination by product quality when quality is only observable to consumers. A seller offers durable goods of uncertain quality. She introduces new inventory at high prices and gradually marks down unsold inventory. Consumers arrive sequentially, choose a price point at which to inspect a good, observe its quality, and decide whether to purchase. Their decisions endogenously sort products by quality across markdown levels, resulting in indirect price discrimination by quality. I characterize sorting equilibria: steady states in which consumers optimally choose prices for inspection and sustain the quality distribution through their purchases. Despite the richness of the equilibrium set, the main result shows that any equilibrium can be summarized by a single statistic. Consumer and seller payoffs depend only on the relative quality difference between the highest and lowest prices, while intermediate markdowns and the sorting path are payoff-irrelevant. I show that the seller faces a fundamental trade-off: she must sacrifice sales volume and efficiency to achieve quality-based price discrimination.

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1 Introduction

Many firms offer unsold inventories at reduced prices through markdowns. A vivid example is Filene's Basement, a Boston retailer that implemented an automated markdown system: if an item remained unsold for twelve days, its price was automatically reduced by 25%. After six more days, the markdown rose to 50%, and after another six days, to 75% of the original price (The New York Times, 1982). Today, similar strategies are widespread and appear in different forms: some firms move unsold goods to outlet stores, while others rely on clearance racks or "special offer" tags. In all cases, consumers search among goods that differ in both price and expected quality, depending on how long they have remained unsold.

Markdowns enable sellers to price-discriminate by product quality, even when that quality is unobservable to them. Markdowns reflect prior demand: unsold goods are more likely to be of lower perceived quality. This mechanism is especially valuable when sellers lack prior demand data (for instance, due to short product life cycles) but face sizable marginal costs. Examples include apparel,¹ furniture, and toys.

This indirect form of price discrimination must be sustained by consumer behavior. For example, suppose consumers choose between a full price and a markdown price. Because marked-down items are more likely to be of low quality, consumers weigh the trade-off between price and expected quality. At the same time, their decisions shape quality composition across price levels. If few consumers choose the full price, the lack of full-price sales is uninformative, and many high-quality goods are marked down, reinforcing demand for a markdown price. This feedback loop between consumer behavior and product sorting is the central focus of the paper.

This paper develops a tractable equilibrium model in which markdown pricing endogenously sorts goods by quality through consumer choice. The model allows for a continuum of prices, but remains tractable due to two modeling choices. First, I restrict attention to the steady states. In a sorting equilibrium, consumers choose prices optimally given the quality composition, and sustain this composition constant over time. Second, I fix the pricing process: unsold goods flow linearly through the price structure at rates that keep inventory constant at each price point.

The main result shows that the entire set of sorting equilibria can be described by a single statistic: the ratio of high-quality shares between the highest and lowest prices. Conditional on this ratio, the exact markdown and sorting paths are irrelevant for both consumer and seller payoffs. I use this dimensionality reduction to solve for the monopolist's optimal

¹For instance, Fisher and Raman (1996) provides a case study of Sport Obermeyer, a sportswear manufacturer that commits to production decisions about two years ahead, with 95% of its products being new designs.

equilibrium, but the irrelevance result applies more broadly to any market structure.

The equilibrium highlights a fundamental trade-off: to enable quality-based price discrimination, the seller must forgo sales volume. Successful sales generate surplus, but forgone sales enable sorting and improve the match between the product's quality and its transaction price. Unlike consumer segmentation, this form of price discrimination necessarily reduces total welfare.

Model. Section 3 models a seller who offers many different durable goods. Some goods are more valuable to consumers, for example, because they are perceived as fashionable. Most of the analysis focuses on a binary case, where each good is either high- or low-quality.² All consumers prefer high-quality goods to low-quality ones. In the baseline model, consumers are homogeneous and value each quality type identically.

The seller manages a continuum of locations occupying a segment between 0 and 1. Locations differ in their prices and the share of high-quality goods in their stock. Within a location, prices are uniform across the goods. Depending on the exact implementation of markdowns, a "location" may correspond to a distinct store, a section within a store, or a price tag. Inventory flows continuously across locations downstream (from 0 to 1), and the flow rates are set so that each location's stock remains constant.³

Consumers arrive sequentially and choose where to inspect the goods. As with locations, consumer strategy admits multiple interpretations. If the different locations are different stores, consumers choose which one to visit. Otherwise, consumers choose how to divide their attention between the locations (*e.g.*, browsing the front of the store versus the clearance rack). Each consumer inspects one good at random: the more attention he pays to a given location, the more likely he is to inspect there. The consumer then draws a random good from that location's stock, learns its quality, and decides whether to purchase at the location's posted price. All consumers are short-lived and exit the market after they make their purchasing decision about the inspected good. The seller does not observe which products are drawn by consumers.

A market outcome is the joint distribution of prices, consumer shopping strategy, and the quality composition across the locations. A sorting equilibrium imposes two restrictions. First, the quality composition must be a steady state sustained by the prices (that determine which quality types are purchased) and the consumer strategy. Second, the consumer strategy must be optimal given the prices and the quality composition. Prices are not pinned down

²Throughout, the "quality" is used to describe a taste shock for a good. This shock is common across consumers: all consumers agree on which goods are fashionable. This differs from contexts where quality reflects observable characteristics known to the seller.

 $^{^{3}}$ An alternative pricing scheme based on product vintage is shown to be equivalent under steady-state restrictions; see Section 5.

by the equilibrium; instead, they parameterize a set of sorting equilibria.

Main Results. Theorem 1 in Section 4.1 characterizes all sorting equilibria: there are two groups of locations, divided by some threshold. Upstream of the threshold, prices are relatively high, and consumers purchase only high-quality goods; downstream, prices are low, and consumers purchase both qualities. As goods move downstream and get more picked over, the share of high-quality items declines gradually.

Theorem 1 also delivers an irrelevance result. To characterize equilibrium payoffs, one need not track the full dynamics of markdowns or sorting paths. Instead, equilibrium payoffs are fully summarized by a single variable: the sorting precision between the locations offering the highest and lowest prices. The sorting precision is measured by the ratio of high-quality shares between those two locations.

The irrelevance result in Theorem 1 greatly reduces the dimensionality of the analysis. It lets me solve the monopolist's problem in Section 4.3. More broadly, the result applies beyond the single-seller setting as it does not rely on the optimality of the prices. It may prove useful in richer strategic environments, e.g. with upstream manufacturers and downstream off-price retailers.

Proposition 1 formalizes the fundamental trade-off in markdown pricing: the greater the sorting precision and price discrimination by quality, the lower the total quantity sold. In the extreme, perfect sorting (offering no markdowns for high-quality goods) yields zero sales. In equilibrium, a greater sorting precision requires more consumers to shop at high-priced locations that are below the threshold. These consumers purchase high-quality goods before they are marked down, but also encounter low-quality goods that they reject, reducing total sales. As sorting becomes more precise, markdowns steepen, and prices upstream of the threshold rise. Higher prices attract consumers because they imply better odds of finding high-quality goods. This positive equilibrium relationship between prices and consumer price choice resembles a Veblen effect, but is driven by quality sorting rather than conspicuous consumption (as in Bagwell and Bernheim (1996)).

Section 4.3 studies the seller-optimal sorting precision. I show in Proposition 2 that the seller chooses greater sorting precision when (i) the value of high-quality products rises, or (ii) the ex-ante share of high-quality goods is sufficiently high and increases. In each of these cases, more of the welfare is destroyed to sharpen price discrimination by quality.

Section 4.4 extends the model to allow for direct disposal. In the baseline, inventory is cleared only through sales. In practice, however, retailers sometimes discard or donate unsold goods.⁴ In the extension of the model, the seller can choose the rate at which the unsold

⁴For instance, Filene's Basement sent unsold goods to charity after thirty days on the shelf (The New York Times, 1982).

products flow out of the final location at 1. Disposal is costly, representing either disposal expenses directly (*e.g.*, transportation or handling) or the replacement cost (marginal costs of production). For any fixed disposal rate, the main insights of Theorem 1 continue to hold. Moreover, Proposition 3 shows that the seller uses only one way of clearing the low-quality items from her stock. When disposal costs are high, the seller relies on outlet locations with low prices to clear unsold inventory; when the costs are low, the seller discards inventory directly to maintain high prices across all locations.

Vintage-Based Markdowns. The disposal extension sets the stage for a comparison with vintage-based markdowns. The baseline model adopts an inventory-driven pricing process. This specification is best suited to spatial implementations of markdowns, such as a flag-ship/outlet retail structure.⁵ It also simplifies the analysis and yields a particularly sharp characterization of sorting equilibria. An alternative approach is a vintage-based pricing process, closer in spirit to Filene's markdown system, in which prices depend explicitly on the goods' age. Section 5 explores how the model's predictions change under this alternative, where all goods are automatically repriced as they age. Theorem 2 shows that, under the steady-state restrictions, both pricing processes yield equivalent equilibrium outcomes.

Extending and Evaluating the Model. The baseline model captures the equilibrium nature of markdowns in the most tractable setup. Below, I outline three extensions that expand the applicability of the model.

First, the baseline version of the model focuses on infinitely many operated locations: consumers divide their attention smoothly across locations. Proposition 4 in Section 6.1 shows that this restriction is without loss. In fact, the seller strictly benefits from operating infinitely many store locations: a finer markdown structure allows the seller to sort products more precisely, with lower sales losses.

Second, in the baseline model with binary quality, the seller never uses both markdowns and direct disposal in the optimum. To make the model more realistic, Section 6.2 extends the model to allow for multiple quality levels. With this quality uncertainty, the seller may use both clearance methods: offer lower prices to clear intermediate-quality items through sales, but dispose of the lowest quality levels herself.

Third, the baseline model assumes homogeneous consumers. Section 6.3 relaxes this assumption and models heterogeneous consumers with varying marginal utility from quality. Equilibrium payoffs in this extension resemble those of the classical monopolistic screening problem of Mussa and Rosen (1978). Unlike the standard screening models, the menus of prices and qualities are not set by the seller but arise endogenously through consumer choice.

 $^{^{5}}$ Agrawal and Smith (2009) points out that the steepest markdowns are often implemented by moving unsold inventories to outlet stores.

Consumers self-sort across locations: earlier locations offer higher prices and better goods, attracting higher types. These consumers absorb high-quality items, shaping the menus that lower types face at downstream locations.

Some important features of real-world markdowns are intentionally omitted in this paper to isolate the consumer's role in product sorting. A notable omission is time-based quality depreciation, such as seasonality. While depreciation can enrich the sorting mechanism,⁶ it also adds pressure to clear inventory quickly.

The model also assumes a fixed pricing structure. This sets a useful benchmark grounded in real-world practice (such as Filene's) and reflects practical constraints of more nuanced pricing strategies.⁷ Still, the optimal design of markdown structures remains an open question. Future work could examine whether linear stock reallocation is optimal or if more flexible strategies could improve efficiency and profitability.

2 Numerical Example

To build intuition, consider a simplified numerical example. It illustrates the core equilibrium interaction between consumers, pricing, and product sorting.

A seller offers many different goods—such as apparel—and operates two locations: a flagship store (regular-priced) and an outlet store (marked-down). At each location, the total stock of goods is one. Each good is either high- or low-quality. When produced, a good is high-quality with probability 7/9 (independently across different goods). Consumers can observe the quality of any good before purchase. A high-quality good is worth 9 to any consumer, while a low-quality one is worth 1.

The two locations differ in their prices and the stock quality composition. Otherwise, they are identical from the consumer's perspective. At the flagship, two-thirds of the goods are high-quality; at the outlet, one-third are. The flagship charges a higher price of 5, while the outlet's price is 1 (the value of low-quality goods). Table 1 summarizes the key characteristics of the two locations.

	Flagship	Outlet
Total stock mass	1	1
Price per item	5	1
Share of high-quality items	2/3	1/3

Table 1: Numerical Example: Flagship and Outlet

⁶For instance, the seller may simply wait for high-value consumers to buy before marking goods down.

⁷For example, Caro, Babio, and Peña (2019) note that Zara releases around 8,000 products per year, making granular pricing decisions too costly.

A unit mass of consumers each chooses which location to visit. Consumers know the prices and quality compositions at both locations before making a decision. Upon visiting, each consumer is randomly matched to a single good from the selected location's stock. That is, the probability of drawing a high-quality item is 2/3 at the flagship and 1/3 at the outlet. Then, the consumer observes the quality of the drawn good, decides whether to purchase it at the (visited location's) posted price, and exits the market. Consumers thus trade off price against the chance of finding a more valuable good. Relative attractiveness of the flagship's quality composition is captured by the *sorting precision*, the ratio of high-quality shares between the flagship and the outlet.

Consumers' trade-off is balanced in this example: they are indifferent between the two locations. At the flagship, consumers only buy when they find a high-quality item, receiving a payoff of 9-5=4. As high-quality items are drawn with probability 2/3, the expected payoff from visiting the flagship is $2/3 \times 4 = 8/3$. At the outlet, consumers buy all items regardless of quality. The expected payoff at the outlet is $1/3 \times (9-1) + 2/3 \times (1-1) = 8/3$, same as at the flagship.

Purchases affect the quality composition differently across locations. Suppose 3/4 of consumers visit the flagship and 1/4 visit the outlet. Since consumers are indifferent between the two locations, each chooses optimally. At the flagship, consumers only buy high-quality goods in the mass of $3/4 \times 2/3 = 1/2$. This leaves 2/3 - 1/2 = 1/6 of the high-quality items unsold. The mass of the remaining low-quality items stays at 1/3. The resulting post-purchases high-quality share in the flagship's stock is

$$\frac{1/6}{1/6 + 1/3} = 1/3.$$

At the outlet, the stock composition is unchanged. Consumers purchase any good they are matched to, regardless of quality. The total mass of purchases equals the total number of outlet visitors 1/4.

The initial inventory levels and quality composition from Table 1 can be restored even if the seller cannot observe product quality. To replenish the outlet, the seller transfers 1/4mass of goods from the flagship (selecting goods at random). Recall that post-purchases, the flagship has the same high-quality share as the outlet, 1/3. The outlet's quality composition is thus restored. The flagship, having transferred 1/4 and sold 1/2, retains a stock of mass 1/4. To restore it to full capacity, the seller adds 3/4 mass of new items. The resulting flagship's high-quality share is:

$$3/4 \times 7/9 + 1/4 \times 1/3 = 2/3.$$

The quality composition is *sustained* by the prices and the consumer shopping strategy. Prices determine which product types are purchased at each location, and consumers' location choices determine the volume of purchases. Together, they pin down the inflows and outflows of high-quality goods and ensure the quality composition stays at a steady state.

Prices, consumer location choice, and quality composition together constitute a sorting equilibrium. In my equilibrium definition, I require that (i) prices and consumer strategy sustain the quality composition, and (ii) consumers choose location optimally. As we have verified, both of these conditions are satisfied in the numerical example. Note that the equilibrium takes prices as exogenously given.

Now consider what happens if the flagship price rises to 7 (while the outlet price stays the same). Table 2 summarizes the resulting new sorting equilibrium, in which each location still serves a positive mass of consumers.

	Flagship	Outlet
Price	7	1
Consumer Share	pprox 0.86	pprox 0.14
Share of high-quality	≈ 0.5	≈ 0.12

Table 2: New Sorting Equilibrium

With a higher flagship price, the sorting precision rises. Under the initial flagship price of 5, the sorting precision was $2/3 \div 1/3 = 2$. As the price increases to 7, the sorting precision doubles. Sharper sorting preserves consumer indifference: while the flagship is now more expensive, it offers even better odds of finding a high-value product relative to the outlet.

Higher sorting precision can be sustained only when more consumers choose to visit the flagship location. They pick out more of the high-quality goods, and the flagship's postpurchases remaining stock is more likely to be of low quality. This generates a positive equilibrium relationship between the flagship price and its customer share. To an outside observer who does not account for the equilibrium sorting effects, it may appear like an upward-sloping demand for the flagship store, or a Veblen effect.

Higher flagship price reduces total surplus and worsens quality composition at both locations. Some consumers who previously visited the outlet now switch to the flagship and only buy if they draw a high-quality product. As a result, total purchases fall. With slower turnover, fewer new items are added, and low-quality goods remain in inventory longer. This lowers the share of high-quality goods in both stores' inventory.

In Appendix OA1, I generalize this equilibrium comparative statics with respect to the flagship price. Figure 1 summarizes the main equilibrium effects. Figure 1a plots how the equilibrium sales volume and the flagship customer share vary with the flagship price. As the

flagship price rises, more consumers visit the flagship location, but total sales fall. Figure 1b shows that the equilibrium share of high-quality goods falls at both stores with the flagship's price. The decline is steeper at the outlet, resulting in greater sorting precision.



Figure 1: Equilibrium Comparative Statics with Respect to Flagship Price

Note: The figure shows how key model outcomes vary with flagship price in the interior sorting equilibrium. Figure 1a plots the volume of flagship customers (dashed) and total per-period sales (solid). Figure 1b shows high-quality shares at the flagship (solid black) and outlet (dashed black), and sorting precision (purple, dot-dashed), defined as their ratio.

What is the seller-optimal sorting equilibrium? The numerical example illustrates a key trade-off: higher sorting precision improves the seller's ability to identify and price high-quality goods, but it comes at the cost of lower total sales. At one extreme, the seller can set both prices to 1, giving up on the product sorting entirely. All consumers purchase their matched item, and the stock composition is at the production plant's level. The seller earns a total profit of 1 each period. At the other extreme, the seller can raise the flagship price to nearly 9, the maximum consumers are willing to pay for high-quality goods. This requires near-perfect sorting but slows inventory turnover so much that both stores become depleted of high-quality goods, and sales collapse. In the optimum, the seller sets some interior price that balances these opposing forces.⁸

⁸For instance, both of the above extremes are dominated by the numerical example of Table 1. To verify, the seller earns a profit of $5 \times 1/2 = 5/2$ at the flagship store by charging a price of 5 from all flagship consumers who get matched to a high-quality product. At the outlet, the seller earns 1/4 by making sales to all its consumers at a price of 1. As a result, the seller achieves a total profit of 11/4 from both locations.

3 Model

The section presents a model of equilibrium markdowns with a rich pricing structure. A seller produces goods of unknown quality and sets prices across a continuum of locations. Consumers choose where to shop based on the anticipated quality distribution, draw one good at random, learn its quality, and decide whether to purchase. Prices determine which products are purchased at each location. The markdown process is fixed: unsold inventory flows downstream and is repriced at rates that keep stock constant across price levels. I introduce sorting equilibrium, which formalizes endogenous product sorting by quality through consumer choice. It imposes steady-state restrictions on prices, consumer behavior, and the quality distribution. The next section characterizes all such equilibria in the main result, Theorem 1, and studies the seller's optimal choice among them.

Products. A single long-lived seller (female) offers durable goods that differ in quality, which is only observable to consumers. The seller perceives all goods as identical, and bears 0 marginal cost of producing either quality.⁹ Consumers (males), by contrast, observe each product's quality before purchase. They derive utility v^h from high-quality products and v^l from low-quality ones, where $v^h > v^l > 0$. For interpretation, the quality of the good reflects if it is fashionable.

Locations. The seller manages a continuum of *locations* indexed by $x \in X = (0, 1)$. Location 0 is the production plant. The total inventory is normalized to 1 and is uniformly distributed over X.¹⁰ Each location $x \in X$ is characterized by its price $\mathbf{p}(x)$ and the share of high-quality products in its stock $\mathbf{q}(x)$. Both the *price schedule* $\mathbf{p} : X \to \mathbb{R}$ and the *quality composition* $\mathbf{q} : [0, 1) \to [0, 1]$ are (Lebesgue-)measurable. The probability of a high-quality product at the production plant, $\mathbf{q}(0)$, is fixed at $\pi \in (0, 1)$. Depending on the application, locations may represent distinct stores (flagship/outlet), parts within the store (storefront/clearance rack), or price tags (regular/marked-down).

Consumers. Time is continuous, and at every instant, a flow of short-lived consumers arrives at the market at a unit rate.¹¹ Continuous arrival of consumers implicitly assumes that the purchases are small relative to the stock at any location.¹²

Consumers allocate a limited amount of attention across the locations. The consumer

⁹This is a normalization. I could alternatively shift consumer values by the constant marginal cost of production. I elaborate on this later, in Section 4.4.

¹⁰For the subsequent analysis, it is enough to assume that the distribution of stock is absolutely continuous with respect to Lebesgue measure on X. The uniform distribution is a normalization.

¹¹Over a time interval of length Δ , the mass of arriving consumers is Δ .

 $^{^{12}}$ To better relate to the two-store example from the introduction, the model in this section is a *double limit* of a discrete model, as I both reduce the per-period mass of consumers and increase the number of equally sized stores.

strategy is a density function $\sigma : X \to \mathbb{R}_+$ that describes the distribution of consumer attention. Each consumer draws a single product at random. His strategy determines where the product is drawn. The more attention a consumer pays to some locations, the more likely he is to draw a product from there. For instance, the probability of drawing a product from an interval $[x_1, x_2]$ is $\int_{x_1}^{x_2} \sigma(y) dy$.¹³ I say that a location x is visited if it gets positive attention from consumers: $\sigma(x) > 0$.

Conditional on the location, a product is drawn at random from its available inventory. If location x holds a share $\mathbf{q}(x)$ of high-quality goods, then the consumer finds a high-quality item at x with probability $\mathbf{q}(x)$. Upon drawing the product at location x, the consumer learns its quality and decides whether to buy it at price $\mathbf{p}(x)$. A consumer earns a payoff $v^{\omega} - p$ when purchasing a product of type $\omega \in \{l, h\}$ at price p. For notational simplicity, ties in the purchasing decisions are broken in favor of a purchase.¹⁴ The seller does not see which products are drawn by the consumers but are not purchased.

Prices fully determine which qualities are purchased at each location. A product of type ω is purchased at location x whenever $v^{\omega} \geq \mathbf{p}(x)$. Then, there are three groups of locations: (*i*) where neither quality is purchased, (*ii*) where only high-quality products are purchased, and (*iii*) where both qualities are purchased. I refer to locations in the third group as outlets. Formally, a location x is an outlet if $\mathbf{p}(x) \leq v^l$. Otherwise, it is a non-outlet location.

The consumer expected payoff depends on the full market outcome $(\mathbf{p}, \sigma, \mathbf{q})$: a tuple of the price schedule, the quality composition, and the consumer strategy. The consumer strategy σ first determines the probability of drawing a product from each location. Conditional on drawing a product from location x, the quality composition \mathbf{q} gives the probability that it is of high quality. Finally, the price \mathbf{p} determines the terms of trade. The consumer expected payoff at the market outcome $(\mathbf{p}, \sigma, \mathbf{q})$ is:¹⁵

$$V^{B}(\mathbf{p},\sigma,\mathbf{q}) = \int_{x \in X} \left[\mathbf{q}(x)(v^{h} - \mathbf{p}(x))_{+} + (1 - \mathbf{q}(x))(v^{l} - \mathbf{p}(x))_{+} \right] \sigma(x) dx.$$

Pricing Process: Product Flows. Inventory moves to and from locations due to two forces: consumer purchases and reallocations within X.¹⁶ Unsold inventory flows linearly in the same direction from 0 to 1: each location receives inventory from its immediate upstream neighbor and passes goods downstream.¹⁷ It follows that the average age of the reallocated

 $^{^{13}{\}rm Equivalently},$ consumers choose a single location to visit. Consumer strategy is then simply their mixing strategy over all locations.

¹⁴This assumption does not play any substantive role in the analysis as it focuses on seller-preferred outcomes.

¹⁵Here, and elsewhere $(a)_+ \equiv \max\{a, 0\}$ for any $a \in \mathbb{R}$.

¹⁶In Section 4.4, I also allow goods to flow from location 1 for disposal.

¹⁷For any two locations, x < y, location is y is downstream of x and location x is upsteam of y.

goods increases with a location's index.

The rates at which the products flow keep each location at its full capacity. Specifically, the inventory flow rate through a location x at the market outcome m equals its *downstream* sales $S_m(x)$, the flow of purchases at all locations that are strictly downstream of x:

$$S_m(x) = \int_{y \in (x,1)} \left[\mathbf{q}(y)\sigma(y) \mathbb{1}\{\mathbf{p}(y) \le v^h\} + (1 - \mathbf{q}(y))\sigma(y) \mathbb{1}\{\mathbf{p}(y) \le v^l\} \right] dy.$$

Say that a location x is a transition location in a market outcome m if it passes a positive flow of goods: $S_m(x) > 0$. The seller does not observe the product quality, or consumer product draws. Conditional on the location, she picks the reallocated inventory at random: high-quality products flow through the location x at a rate $\mathbf{q}(x)S_m(x)$.

As this reallocation process is inventory-based, it best applies to spatial implementations of markdowns, such as a flagship/outlet retail structure. This process makes the relationship between sorting, prices, and sales the most transparent. It directly links the rate of price changes to the rate of (downstream) sales. However, it does not allow the seller to price the goods based on their age directly. In Section 5, I consider another, vintage-based pricing process. I show that the two yield the same equilibrium outcomes when we narrow the analysis to the steady states.

I summarize the inventory flows in Figure 2. Fix an interval of locations $(x_1, x_2] \subseteq X$, and consider all inventory flows from and to this interval. First, inventory exits this interval due to consumer purchases. The total mass of high-quality purchases within the interval is:

$$\int_{y \in (x_1, x_2]} \mathbf{q}(y) \sigma(y) \mathbb{1}\{\mathbf{p}(y) \le v^h\} dy,$$

and the total mass of low-quality purchases is:

$$\int_{y \in (x_1, x_2]} (1 - \mathbf{q}(y)) \sigma(y) \mathbb{1}\{\mathbf{p}(y) \le v^l\} dy.$$

In addition to purchases, inventory exits the interval through its right boundary x_2 , which forwards products to downstream locations.¹⁸ The total mass reallocated downstream from x_2 is $S_m(x_2)$, of which mass $\mathbf{q}(x_2)S_m(x_2)$ is high-quality. The interval receives inventory exclusively from location x_1 . The total inflow is $S_m(x_1)$, and the inflow of high-quality goods is $\mathbf{q}(x_1)S_m(x_1)$.

¹⁸Interior locations (x_1, x_2) also pass goods downstream, but these shipments only redistribute inventory within the interval (x_1, x_2) and do not affect its inflows or outflows.



Figure 2: Product Flows within a Period

The mass of inventory is kept constant by construction, but the quality composition within an interval $(x_1, x_2]$ may change over time. It only stays constant when inflows and outflows of high-quality goods are balanced:

$$\int_{y \in (x_1, x_2]} \mathbf{q}(y) \sigma(y) \mathbb{1}\{\mathbf{p}(y) \le v^h\} dy + \mathbf{q}(x_2) S_m(x_2) = \mathbf{q}(x_1) S_m(x_1).$$
(1)

We say that the quality composition \mathbf{q} is sustained on $A \subseteq X$ by prices and consumer strategy (\mathbf{p}, σ) if Equation (1) holds for any interval $(x_1, x_2] \subseteq A$. That is, the average quality of products is at a steady state for any subinterval of A. For brevity, \mathbf{q} is sustained by (\mathbf{p}, σ) if it is sustained on X.

Sorting Equilibrium. The central goal of the model is to capture the interdependence between prices, consumer choices, and quality composition. To capture it in a tractable way, I use an equilibrium concept that imposes steady-state restrictions on the market outcomes. A market outcome $(\mathbf{p}, \sigma, \mathbf{q})$ is a sorting equilibrium if:

- (i) the quality composition \mathbf{q} is sustained by (\mathbf{p}, σ) ;
- (ii) each visited location maximizes consumer payoff given (\mathbf{p}, \mathbf{q}) :

$$x \in \underset{y \in X}{\operatorname{arg\,max}} \mathbf{q}(y)(v^{h} - \mathbf{p}(y))_{+} + (1 - \mathbf{q}(y))(v^{l} - \mathbf{p}(y))_{+}.$$

Let \mathcal{E} denote the set of sorting equilibria.

The sorting equilibrium captures how consumer beliefs about the quality composition become self-fulfilling through optimal choices. Suppose consumers follow a time-invariant strategy σ , then the quality composition at every period t is described by:¹⁹

$$\partial_t \mathbf{q}_t(x) = -\sigma(x)\mathbf{q}_t(x)\mathbb{1}\{\mathbf{p}(x) \le v^h\} - \partial_x \left[S_{m_t}(x)\mathbf{q}_t(x)\right],$$
$$\mathbf{q}_t(0) = \pi.$$

A sustained quality composition ${\bf q}$ is a time-invariant solution to the above, capturing the long-run limit.²⁰

To justify a time-invariant consumer strategy, assume consumers do not observe either the calendar date or the current quality composition. Then, condition (ii) restricts consumer beliefs: at any date, consumer beliefs are dominated by the true long-run quality composition. **Seller**. The seller maximizes her long-run profit flow

$$V^{S}(\mathbf{p},\sigma,\mathbf{q}) = \int_{\mathbf{p}(x)\in(v^{l},v^{h}]} \mathbf{p}(x)\mathbf{q}(x)\sigma(x)dx + \int_{\mathbf{p}(x)\leq v^{l}} \mathbf{p}(x)\sigma(x)dx$$

by selecting a sorting equilibrium:

$$\sup_{(\mathbf{p},\sigma,\mathbf{q})\in\mathcal{E}} V^S(\mathbf{p},\sigma,\mathbf{q}).$$

To interpret, the seller posts prices and then nudges consumers towards self-fulfilling beliefs about the quality composition. The steady-state restriction can be viewed as reflecting the seller's aversion to fluctuations in long-run profit. Methodologically, it greatly simplifies the analysis: an intrinsically dynamic sorting process can be studied in a static environment. While this approach rules out many potential seller strategies, the steady-state equilibrium model provides a useful and compelling benchmark.

Efficient Benchmark. As a benchmark, suppose there is no seller, and consumers draw their goods directly from production. The total surplus equals the expected value of a random new good: $\pi v^h + (1 - \pi)v^l$. This is the efficient benchmark. The seller could extract this surplus if she observed which quality type is drawn by every consumer.

Formally, let the total surplus of a market outcome $(\mathbf{p}, \sigma, \mathbf{q})$ be the sum of the seller's

¹⁹Where $m_t = (\mathbf{p}, \sigma, \mathbf{q}_t)$

²⁰To make this argument crisper, one would need to show convergence of \mathbf{q}_t to the steady state. In Appendix C, I show convergence in simulations. In Online Appendix OA1, I verify convergence for a simpler version of the model with two stores used for the numerical example.

flow profit and consumer payoff:

$$TS(\mathbf{p}, \sigma, \mathbf{q}) = V^S(\mathbf{p}, \sigma, \mathbf{q}) + V^B(\mathbf{p}, \sigma, \mathbf{q}).$$

A market outcome is efficient if its total surplus achieves the efficient benchmark of $\pi v^h + (1 - \pi)v^l$.

Different Paths, Same Destination. In general, the seller's problem is very rich: complementarities in consumer choices imply multiplicity of sorting equilibria. Figure 3 plots two sorting equilibria. In the first sorting equilibrium $(\mathbf{p}_1, \sigma_1, \mathbf{q}_1)$, consumers pay equal attention to all locations. Under the second one, $(\mathbf{p}_2, \sigma_2, \mathbf{q}_2)$, consumer attention gets polarized towards the extremes.

The two plotted equilibria take different sorting paths from the production plant's expected quality π towards the quality composition q_o at their outlet locations. Which one should the seller choose among these two options? In the next section, I show that they are payoff equivalent.



Figure 3: Two Examples of the Sorting Equilibrium

Note: the figure illustrates two sorting equilibria $(\mathbf{p}_1, \sigma_1, \mathbf{q}_1)$ (solid lines) and $(\mathbf{p}_2, \sigma_2, \mathbf{q}_2)$ (dashed lines). The two equilibria are different in their paths of pricing and sorting, but have the same overall sorting precision of π/q_o .

4 Sorting Equilibria

This section analyzes the model. First, section 4.1 characterizes the set of sorting equilibria, the feasibility set of the seller. Theorem 1 shows an irrelevance result: only the overall sorting precision between the highest and the lowest prices matters for equilibrium payoffs, not specific sorting and pricing paths. I use this dimensionality reduction to analyze the seller's problem in Section 4.3. Section 4.4 extends the model to allow for direct disposal of

goods.

4.1 Equilibria Characterization: Irrelevance Result

Theorem 1 establishes two key properties of the sorting equilibria. First, every sorting equilibrium features an *outlet threshold* $\hat{x} \in [0, 1]$: visited locations upstream of \hat{x} charge prices above v^l , while all visited locations downstream of \hat{x} are outlets. Generally, say that a market outcome is a \hat{x} -threshold market outcome if $\mathbf{p}(\cdot) > v^l$ on $(0, \hat{x})$ (σ -a.s.)²¹ and $\mathbf{p}(\cdot) \leq v^l$ on $[\hat{x}, 1)$ (σ -a.s.).

Second, equilibrium payoffs for both the seller and consumers depend only on the quality composition at the outlet threshold, $\mathbf{q}(\hat{x})$. This parameter also captures the extent of product sorting: the *sorting precision* is defined as the ratio of high-quality shares between the production plant and the outlet threshold, $\pi/\mathbf{q}(\hat{x})$.²² I call a sorting equilibrium *neutral* if $\mathbf{q}(\hat{x}) = \pi$, and *active* if $\mathbf{q}(\hat{x}) < \pi$.

Theorem 1. Suppose a market outcome $(\mathbf{p}, \sigma, \mathbf{q})$ is a sorting equilibrium. Then, it is a \hat{x} -threshold market outcome for some $\hat{x} \in [0, 1]$. Furthermore:

- (i) If no consumers visit outlet locations, i.e., $\int_{\hat{x}}^{1} \sigma(y) dy = 0$, then both consumer and seller payoffs are zero.
- (ii) If (almost) all consumers visit outlet locations, i.e., $\int_{\hat{x}}^{1} \sigma(y) dy = 1$, then the sorting equilibrium is neutral and efficient. The seller's payoff is at most v^{l} .
- (iii) If consumers visit both outlet and non-outlet locations, i.e., $\int_{\hat{x}}^{1} \sigma(y) dy \in (0, 1)$, then the sorting equilibrium is active and inefficient. The payoffs are fully determined by the quality composition at the outlet threshold \hat{x} , with:

$$TS(\mathbf{p}, \sigma, \mathbf{q}) = \frac{\pi v^h + (1 - \pi)v^l}{\ln\left(\frac{\pi}{1 - \pi}\frac{1 - \mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right)(1 - \pi) + 1}$$
$$V^B(\mathbf{p}, \sigma, \mathbf{q}) = \mathbf{q}(\hat{x})(v^h - v^l).$$

Finally, for any $q \in (0, \pi]$, there exists a sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q}) \in \mathcal{E}$ that has average quality q at the outlet threshold \hat{x} : $\mathbf{q}(\hat{x}) = q$.

²¹That is, $\int_{\mathbf{p}(y) \leq v^l, y \in (0,\hat{x})} \sigma(y) dy = 0$. More generally, I say that a statement A holds σ -a.s. if the measure of visited locations where A is false is 0: $\int_{y \in X, \text{where } \neg A} \sigma(y) dy = 0$.

²²Alternatively, we could define the sorting precision as the ratio between the earliest visited location $x_0 = \sup\{x \in X : \int_0^x \sigma(x) = 0\}$ and the outlet threshold \hat{x} . This would be a more direct analog to a sorting precision in the numerical two-store example. However, by Lemma 3, $\mathbf{q}(x_0) = \pi$ in every sorting equilibrium, so the two are equivalent.

Theorem 1 divides all sorting equilibria into three types, based on how consumers distribute their attention between outlet and non-outlet locations.

If no consumers shop at outlet locations (part (i)), sales collapse entirely, and both the seller and consumers receive zero payoffs. Without outlets, low-quality goods pile up, ultimately crowding out all sales at high prices. If consumers only visit outlets (part (ii)), the quality composition remains constant at π , and all consumers buy the goods they draw. There is no sorting, but also no inefficiency.

In the most interesting case, consumers visit both outlet and non-outlet locations (part (iii)). The sorting equilibrium is active and inefficient, with non-zero sales. Part (iii) also formulates the irrelevance result. Once the sorting precision $(\pi/\mathbf{q}(\hat{x}))$ is fixed, other details of prices, consumer strategy, or sorting are payoff-irrelevant. While the main focus of the subsequent analysis is a monopolist's problem, the irrelevance result applies to any market structure. The sorting equilibrium does not require prices to be optimal: they could be set by decentralized sellers. Theorem 1 closes the characterization of attainable payoffs by verifying that any sorting precision can be achieved in some sorting equilibrium.

The irrelevance result lets us easily rank payoffs in all sorting equilibria with positive sales.

Corollary 1. Consider two sorting equilibria $m_1 = (\mathbf{p}_1, \sigma_1, \mathbf{q}_1), m_2 = (\mathbf{p}_2, \sigma_2, \mathbf{q}_2) \in \mathcal{E}$ with outlet-thresholds $\hat{x}_1, \hat{x}_2 < 1$, respectively. If $\mathbf{q}_1(\hat{x}_1) > \mathbf{q}_2(\hat{x}_2)$, then both total surplus and consumer payoff in m_1 are higher than in m_2 .

Figure 4 plots all attainable equilibrium payoff pairs (V^B, V^S) . Greater sorting precision shrinks total surplus but reallocates more of it to the seller by bringing transaction prices for high-quality goods closer to the consumers' willingness to pay. Depending on the strength of each effect, the seller's payoff may either increase or decrease. I return to this in Section 4.3, when I study the seller-optimal sorting precision.

Figure 4 highlights that, unlike consumer segmentation, price-discrimination by product quality necessarily destroys total surplus. When screening consumer types, non-uniform pricing can sometimes improve efficiency (e.g., by improving the match between the product quality and the buyer's willingness to pay). When the seller relies on non-sales to price-discriminate her goods, any price-discrimination is wasteful.



Figure 4: Attainable Equilibrium Payoffs

4.2 Proof Sketch of Theorem 1

The remainder of the section outlines the main steps in the proof of Theorem 1 and illustrates how sorting equilibria are constructed. Formal proofs and technical details are deferred to the Appendices.

Sorting Process. The starting point is to understand how prices and consumer strategies shape the quality composition across locations. Lemma 1 summarizes the key restrictions on **q** that can be sustained by some prices and consumer strategy (\mathbf{p}, σ) on an interval. It considers two cases, depending on whether the locations in the interval are outlets.

Lemma 1. Consider some market outcome $m = (\mathbf{p}, \sigma, \mathbf{q})$.

- i) Suppose that $\mathbf{p}(\cdot) > v^l \ (\sigma\text{-a.s.})$ over $[x_1, x_2] \subset X$. Then, \mathbf{q} is sustained by (\mathbf{p}, σ) on $[x_1, x_2]$ if and only if $S_m(x)(1 \mathbf{q}(x))$ is constant over $[x_1, x_2]$.
- ii) Suppose that $\mathbf{p}(\cdot) \leq v^l \ (\sigma\text{-}a.s.)$ over $[x_1, x_2] \subset X$ and x_2 is a transit location. Then, \mathbf{q} is sustained by (\mathbf{p}, σ) on $[x_1, x_2]$ if and only if $\mathbf{q}(x)$ is constant over $[x_1, x_2]$.

Proof. See Appendix C for a proof.

In words, Lemma 1 states that at non-outlet locations, the downstream reallocation of low-quality goods is constant. This ensures that inflows and outflows of low-quality goods balance, as these goods only move through downstream reallocations at non-outlet locations.

At outlets, no sorting occurs, and the quality composition stays constant. Intuitively, as consumers purchase both quality types, there is no information revealed by the lack of sales. Lemma 1 implies that all sorting by quality occurs at non-outlet locations, and the likelihood of high-quality goods declines at the rate of $\sigma(\cdot)/S_m(\cdot)$:²³

$$\partial_x \mathbf{q}(x) = -\mathbf{q}(x)(1 - \mathbf{q}(x))\frac{\sigma(x)}{S_m(x)}.$$
(2)

Equation (2) shows that sorting is stronger when many consumers visit a location ($\sigma(x) \uparrow$) and when inventory turns over slowly ($S_m(x) \downarrow$). Both of these strengthen the adverse selection effect on the remaining inventory.

No Outlets: Sales Collapse. If no outlets are visited, the sales collapse.

In the absence of low-priced sales, the low-quality items gradually fill all available shelf space. Consequently, consumers can not find any high-quality goods that are worth purchasing at the high-priced locations. As the seller makes no sales, both the seller and consumers receive a zero payoff. This corresponds to a \hat{x} -market outcome for the outlet threshold $\hat{x} = 1$, in which the seller sorts the goods perfectly,²⁴ but sorting backfires and chokes off all sales. **Threshold Structure**. Any sorting equilibrium with positive sales is a \hat{x} -threshold market outcome.

This threshold structure arises because the quality composition $\mathbf{q}(x)$ is non-increasing (by Lemma 1). Then, if consumers visit \hat{x} , they only visit locations downstream of \hat{x} if they offer (weakly) lower prices.

Only Outlets: Neutral Equilibrium. Moving to part (*ii*) of the theorem, assume that all visited locations are outlets. By Lemma 1 part (*ii*), quality composition is constant over all (transit) locations. Since new inventory arrives with quality π , the quality composition at all locations must also be π .²⁵ Thus, the sorting equilibrium is neutral. As all consumers make a purchase, the sorting equilibrium is also efficient. Given that the prices at all visited locations are at most v^l by the premise, the seller earns at most v^l .

Active Sorting: Irrelevance Result. Finally, when consumers visit both outlet and nonoutlet locations, there is active sorting of products. I now illustrate why the payoff structure is determined entirely by the quality at the outlet threshold $\mathbf{q}(\hat{x})$.

Consumer Payoff. If consumers visit both location types, their payoff is $V^B(\mathbf{p},\sigma,\mathbf{q}) =$

$$\partial_x((1-\mathbf{q}(x))S_m(x)) = -S_m(x)\partial_x\mathbf{q}(x) - (1-\mathbf{q}(x))\mathbf{q}(x)\sigma(x) = 0.$$

Equation (2) follows.

²⁴As outlets are not visited, they must have zero high-quality goods, or else consumers could obtain positive payoff there.

 $[\]overline{ ^{23}$ To derive Equation (2), we differentiate $(1-\mathbf{q}(x))S_m(x)$. From Lemma 1 part (i), $\partial_x((1-\mathbf{q}(x))S_m(x)) = 0$ (a.e.) on $[x_1, x_2]$. Consumers only make a purchase when they draw high-quality goods (a.e.) on $[x_1, x_2]$: $\partial_x S_m(x) = -\sigma(x)\mathbf{q}(x)$. Together, these two imply:

²⁵Formally, in the Appendix Lemma 3 shows that \mathbf{q} is continuous at every transit location.

 $\mathbf{q}(\hat{x})(v^h - v^l).$

The proof of this statement relies on consumer indifference between all visited locations. Note that the outlet locations (by their definition) guarantee consumers a payoff of at least $\mathbf{q}(\hat{x})(v^h - v^l)$.

On the other hand, as the quality composition is continuous on $[0, \hat{x}]$ (see Lemma 3), consumers must visit some high-priced locations (upstream of \hat{x}), whose quality composition is arbitrarily close to $\mathbf{q}(\hat{x})$. But then the price at the outlet locations cannot be lower than v^l . Otherwise, there are some visited locations, where the quality is only marginally better, but where the price is discontinuously higher. This violates consumer optimality. Then, the only consumer payoff that is consistent with consumer optimality is exactly $\mathbf{q}(\hat{x})(v^h - v^l)$. **Total Surplus**. The quality composition at the outlet threshold also determines the total surplus in a sorting equilibrium. I show this in two steps. First, Lemma 2 shows that the

total surplus in a sorting equilibrium is as if only new products are drawn, but corrected for the actual sales volume. Second, Proposition 1 establishes a monotone relationship between the sorting precision and the sales volume.

Lemma 2. If $m = (\mathbf{p}, \sigma, \mathbf{q})$ is a sorting equilibrium, then the total surplus at m is given by:

$$TS(\mathbf{p},\sigma,\mathbf{q}) = \left(\pi v^h + (1-\pi)v^l\right)S_m(0) \tag{3}$$

To prove Lemma 2, consider the total surplus generated at outlet and non-outlet locations separately. At outlet locations $[\hat{x}, 1)$, quality composition is constant at $\mathbf{q}(\hat{x})$ (σ -a.s.) by Lemma 1, and consumers buy both product types. Thus, the total surplus generated at outlets equals:

$$\left[\mathbf{q}(\hat{x})v^h + (1 - \mathbf{q}(\hat{x}))v^l\right]S_m(\hat{x}).$$

At non-outlet locations $(0, \hat{x})$, consumers only purchase high-quality goods. Thus, they generate the surplus of

$$v^h(S_m(0) - S_m(\hat{x})).$$

Summing these, we obtain that the total surplus generated on X is:

$$TS(\mathbf{p}, \sigma, \mathbf{q}) = v^{h}(S_{m}(0) - S_{m}(\hat{x})) + \left[\mathbf{q}(\hat{x})v^{h} + (1 - \mathbf{q}(\hat{x}))v^{l}\right]S_{m}(\hat{x})$$

= $v^{h}S_{m}(0) - (v^{h} - v^{l})(1 - \mathbf{q}(\hat{x}))S_{m}(\hat{x}).$

Finally, using Lemma 1, we may replace $(1 - \mathbf{q}(\hat{x}))S_m(\hat{x})$ in the above with $(1 - \pi)S_m(0)$, which completes the proof of the lemma.

The proof uncovers the main reasons for the irrelevance result: sales generate both surplus

and sorting of products. Consequently, any sorting precision is associated with a particular sales volume. It does not matter what exact path the sorting takes, as in equilibrium, the sales rate adjusts accordingly.

Sales Volume and Sorting Precision. The final step is to derive the equilibrium relationship between the sorting precision and total sales volume.

Proposition 1. Consider a \hat{x} -threshold market outcome $m = (\mathbf{q}, \sigma, \mathbf{p})$ with positive total sales $S_m(0)$. If m is a sorting equilibrium, then:

$$S_m(0) \left[\ln \left(\frac{\pi}{1 - \pi} \frac{1 - \mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})} \right) (1 - \pi) + 1 \right] = 1.$$
 (Q-S)

Proof. See Appendix C.

Proposition 1 shows that the more precise the sorting (lower $\mathbf{q}(\hat{x})$), the lower the total number of sales. Figure 5a illustrates two sources of the sales loss as the sorting precision increases. It compares two sorting equilibria with different sorting precisions (but the same uniform consumer strategy).

As sorting becomes more precise, the outlet threshold moves downstream from \hat{x}_1 to \hat{x}_2 , increasing the share of consumers visiting high-priced locations. The resulting drop in sales has two components. The first effect is *direct* (dotted area on Figure 5a): consumers drawing goods in (\hat{x}, \hat{x}_2) previously purchased both product types at low prices. Now, they only purchase if they find high-quality goods.

Inventory is renewed more slowly with lower sales volume, and we get a second effect: The quality composition worsens at all locations. Finding high-quality goods at high prices gets more difficult, and the total sales decline further. I refer to this second effect as *the quality-composition effect* (diagonally hatched area on Figure 5a).

Figure 5b shows that a greater sorting precision increases prices at non-outlet locations. Like in the two-store example, we obtain an upward relationship between the prices charged at non-outlet locations and their customer share, which resembles a Veblen effect. Price discrimination by quality strengthens: conditional on sale, the price of the high-quality good becomes higher.



Figure 5: Effects of Greater Sorting Precision

Note: The figure compares two sorting equilibria with different sorting precisions. The two equilibria have the same consumer strategy $\sigma = 1$, but different quality composition and prices. Figure 5a illustrates the effects on the quality composition. By decreasing the share of outlet attention (moving from \hat{x}_1 to \hat{x}_2), the sorting precision increases. The quality composition worsens across all locations. The shaded area shows the resulting total sales loss: the dotted hatch represents the direct effect (fewer purchases at higher prices), and the diagonal hatch captures the quality-composition effect (slower inventory turnover). Figure 5b illustrates the price schedules in the two sorting equilibria.

4.3 Seller-Optimal Sorting

In this section, I solve the seller's problem using the dimensionality reduction from Theorem 1. I show that the seller-optimal sorting precision increases with the value of high-quality goods, or their frequency at the production plant.

By Theorem 1, the seller's problem reduces to selecting the sorting precision. Corollary 2 replaces the seller's objective with $\tilde{V}^{S}(\cdot)$:

$$\tilde{V}^{S}(q) \equiv \frac{\pi v^{h} + (1 - \pi)v^{l}}{\ln\left(\frac{\pi}{1 - \pi}\frac{1 - q}{q}\right) + 1} - q(v^{h} - v^{l}),$$

derived as a difference between the total surplus and consumer payoff.

Corollary 2. A sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q})$ is a solution to the seller's problem if and only if its outlet threshold $\mathbf{q}(\hat{x})$ solves:

$$\max_{q \in (0,\pi]} \tilde{V}^S(q)$$

When choosing the optimal sorting precision, the seller trades off inefficiency due to lost sales against the ability to extract consumer surplus. The negative of the derivative, $-\partial_q \tilde{V}^S(q)$, quantifies the marginal cost and benefit of increasing sorting precision (*i.e.*, lowering q). The first term reflects the rising inefficiency from lower sales volume; the second captures the seller's gain from extracting more consumer surplus.

$$-\partial_q \tilde{V}^S(q) = \overbrace{-\frac{\pi v^h + (1-\pi)v^l}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + 1\right)^2} \frac{1-\pi}{(1-q)q}}^{\text{Inefficiency}} + \underbrace{(v^h - v^l)}_{\text{CS Extraction}}$$

Figure 6 illustrates these two opposing forces. Sorting is costly at either extreme. Near $q \approx \pi$, sorting yields only second-order gains in price but first-order losses in efficiency. Near $q \approx 0$, the sorting becomes nearly perfect and causes sales to collapse. In the interior solution q^* , the costs and benefits of sorting are balanced.

Active sorting is not always optimal. The seller may instead post a single low price v^l for all goods. If active sorting is optimal, then it is quite substantial: $q^* \leq 0.5\pi$ (follows from Lemma 11 in Appendix E). However, the seller never chooses the extreme of pricing all goods at v^h , since that yields no sales.



Figure 6: Marginal Effects of Higher Sorting Precision

Note: the figure capture the marginal effects of product sorting. As product sorting rises (q decreases), the seller benefits by capturing more consumer surplus. At the same time, the seller bears the sorting costs due to inefficiency from foregone sales. If active sorting is optimally, then at the optimal outlet quality composition q^* , the sorting costs cross the sorting benefits from above.

With this trade-off structure in mind, I analyze how the seller's optimal sorting precision

responds to changes in model parameters.

Proposition 2. The optimal quality composition at the outlet threshold decreases if

- (i) v^h increases,
- (ii) or $\pi \approx 1$ and increases.

Proof. See Appendix \mathbf{E} .

Proposition 2 states that the optimal sorting precision rises when either high-quality items become more valuable (part (i)) or very common (part (ii)). Part (i) is very intuitive: as consumers value high-quality items more, the seller wants to raise prices. To do so, the seller must aggravate the price/quality trade-off for the consumer by making sorting more precise.

Part (*ii*) of Proposition 2 is slightly more nuanced. When the probability of high-quality rises, two effects compete. On the one hand, sorting becomes cheaper as finding high-quality goods is relatively easy. On the other hand, it also becomes more costly in terms of the total surplus lost. The first effect dominates when π is sufficiently high.

Together with Proposition 1, these results also imply that the total sales volume falls and welfare losses rise in both cases listed in Proposition 2. Additionally, Proposition 2 (*ii*) implies that the consumer's payoff is decreasing in π when π is high.

Corollary 3. The consumer's payoff at a seller's optimal market outcome is decreasing in π when $\pi \approx 1$.

Intuitively, the seller extracts more consumer surplus from the buyer when the information asymmetry decreases.

4.4 Direct Disposal Extension

This section extends the baseline model by allowing for the direct disposal of unsold inventory. This extension captures a realistic alternative to markdowns for clearing low-quality goods. It also lays the groundwork for comparing the baseline model to an alternative, vintage-based, markdown process in Section 5. Proposition 3 shows that the seller uses only one channel to clear low-quality goods from the stock in the optimum. She offers outlets if disposal costs are high; otherwise, she disposes directly.

Assume the seller may destroy unsold inventory at some constant rate γ . The seller bears per-unit disposal cost $\kappa > 0$. The interpretation of this cost is twofold. Either it is a literal disposal cost, *e.g.*, due to handling or transportation, or it is a per-unit production cost. Under the latter interpretation, the value v^{ω} of a product of type $\omega \in \{h, l\}$ is the consumer's valuation *net* of production costs.

The products are sent for disposal from the location at 1.²⁶ This assumption allows us to preserve the linear nature of the inventory reallocations. The overall structure of the model stays the same, but location 1 now makes special "sales" at a negative price of $-\kappa$. We extend the definition of the sorting equilibrium appropriately. As products now also move due to disposal, the quality composition **q** in a sorting equilibrium ($\mathbf{p}, \sigma, \mathbf{q}, \gamma$) must now sustained be sustained by ($\mathbf{p}, \sigma, \gamma$).

Proposition 3 summarizes the main result for this version of the model: the seller clears low-quality goods from the stock through one channel only.

Proposition 3. The seller clears low-quality goods from the stock through one channel only. In particular for each (v^h, v^l, π) , there exists a threshold disposal cost

$$\bar{\kappa} \in \left(\max_{q \in (0,\pi]} \tilde{V}^S(q) - v^l, \frac{\pi}{1-\pi} v^h\right),\,$$

such that:

(i) if $\kappa < \bar{\kappa}$, then in any optimal sorting equilibrium, there are no outlets,

(ii) if $\kappa > \bar{\kappa}$, then in any optimal sorting equilibrium, there is no direct disposal, i.e., $\gamma = 0$. In addition, as the value of a low-quality product v^l increases, $\bar{\kappa}$ decreases.

Proof. See Appendix F.

In words, the seller chooses the cheapest way to clear low-quality items from the stock. If the disposal cost is relatively low, the seller destroys (some) of the unsold items. This lets the seller maintain uniformly high prices (at v^h) across all locations. Conversely, if disposal costs are high, the seller prefers to offer outlets and delegate the removal of low-quality products to consumers. The bounds on $\bar{\kappa}$ imply sufficient conditions for when each channel is preferred.

When low-quality items are more valuable, the seller is more likely to have low-priced locations. To illustrate, consider the extreme case where $v^l \to v^h$. Then, the salvage value of any product is too high, and the seller does not destroy any goods.

Proposition 3 also clarifies the types of products for which the model is most applicable. Markdowns are most relevant for products that: (i) have highly uncertain demand; (ii) are not easily scalable, so that the seller uses sales information for markdowns but not to adjust production; and (iii) have sizable marginal costs, which make direct disposal less attractive than clearance sales.

 $^{^{26}}$ For simplicity, location 1 is not available for visiting by consumers.

5 Vintage-Based Pricing

This section introduces a vintage-based markdown process, where a product's price depends directly on its time in inventory. Despite the change in pricing mechanics, vintage-based pricing yields the same steady-state equilibrium outcomes as the benchmark model.

As argued above, the pricing process in the baseline model is inventory-driven, and best applies to spatial implementations of markdowns. An alternative markdown process, that mirrors Filene's strategy more closely, is based on the current product's vintage. To model this, I assume that the seller has a limited time horizon for selling each unit of her goods. If the product is not sold by reaching this deadline, the seller must dispose of it directly at a per-unit cost $\kappa > 0$. Since I don't have time depreciation in my model, we can normalize this deadline to 1: $X \in [0, 1]$ now represents the vintage of the goods in the seller's stock. Consumers' choice remains the same: only now they split their attention between different product vintages.

The key difference between the two models lies in how inventory is distributed. Unlike in the benchmark model, where each location holds a fixed number of inventory, the number of unsold goods now varies across vintages. Let $\mu : X \to \mathbb{R}_+$ denote inventory density across vintages. A vintage-based market outcome is a tuple $(\mathbf{p}, \sigma, \mu, \mathbf{q})$ consisting of prices, consumer strategy, inventory distribution, and quality composition.

We adjust the sorting equilibrium definition accordingly to account for the endogeneity of stock distribution across vintages. A vintage-based sorting equilibrium is a market outcome $(\mathbf{p}, \sigma, \mu, \mathbf{q})$ such that: (i) prices and consumer strategy (\mathbf{p}, σ) sustain the inventory distribution μ and quality composition \mathbf{q} ; and (ii) σ maximizes consumer payoff given (\mathbf{p}, \mathbf{q}) . For formal details, see Appendix G.

Theorem 2 establishes equivalence of the two models.

Theorem 2. $(\mathbf{p}, \mu, \sigma, \mathbf{q})$ is a vintage-based sorting equilibrium if and only if $(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ is a sorting equilibrium for the disposal rate $\gamma = \mu(1)$.

Proof. See Appendix G.

The equivalence follows from observing that products of any vintage x are either sold to consumers or disposed of. But then the mass of the vintages x is the same as the downstream sales at x in the original model.²⁷

In both models, goods get repriced at the rate of downstream sales. In the baseline model, these rates of price changes were imposed exogenously. Under vintage-based pricing,

 $^{^{27}}$ After we include the special sales at a negative price at location 1, as discussed in the previous section.

it emerges endogenously due to the steady-state restrictions of the equilibrium. Consequently, we get the exact same predictions for the sustained quality compositions.

6 Extensions and Discussion

This section discusses the limitations of the baseline model and introduces three extensions that address some of them. Section 6.1 examines whether the seller can achieve her optimal payoff with only finitely many visited locations, allowing for more general consumer strategies with atoms. Section 6.2 introduces multiple quality levels to capture richer forms of demand uncertainty. This extension reconciles the coexistence of markdowns and disposal, which the binary-quality model rules out. Section 6.3 incorporates heterogeneous consumers with vertically differentiated preferences for quality. For brevity, formal details of these extensions are presented in the Appendices.

6.1 Atoms in the Consumer Strategy

In this section, I generalize the model with disposal to richer shopping strategies. I show that the seller strictly benefits from operating an infinite number of locations.

Let the consumer strategy be described by a cdf $D : [0,1] \rightarrow [0,1]$, where D(x) denotes the share of consumers who draw products in (0, x]. Unlike the baseline model, D may now have atoms, allowing some locations to attract discrete mass of attention.

We may now investigate whether the seller must operate so many locations to pricediscriminate by binary quality. Proposition 4 shows that the seller cannot achieve her optimal payoff if she strictly prefers active sorting equilibria (to neutral sorting).

Proposition 4. If D admits finitely many discontinuities at non-outlet locations, then the sorting equilibrium $(\mathbf{p}, D, \mathbf{q}, \gamma)$ is suboptimal for the seller.

In words, it is unprofitable for the seller to have a high-priced location with a large customer share. Intuitively, if there is an atom at some non-outlet location, the seller's learning is "bunched". She misses some of the sorting and repricing opportunities.

To illustrate, suppose only two locations are visited: x = 1/3 and x = 2/3, with the latter an outlet. In this case, sorting fails entirely: the average quality is the same at both locations. This reflects a property of the continuous-time model: consumer purchases are infinitesimal relative to inventory, so that there can be no sorting within a single store at

 $1/3.^{28}$

This result underscores another distinction between price discrimination by quality and monopolistic screening. When screening consumer types, the seller only needs one menu item per consumer type. In contrast, when sorting product types, the seller prefers a continuum of locations, even if product quality is binary.

6.2 Multiple Quality Levels

Section 4.4 shows that in the binary quality model, direct disposal and markdowns are mutually exclusive: the seller clears low-quality goods using only one method. To better reflect real-world practices, this section extends the model to allow for products with multiple quality levels.

Suppose the product comes in n quality levels, yielding consumer values $v^1 < v^2 < \cdots < v^n$. As in the baseline model, consumer strategy admits a density over X. The quality composition is now defined as $\mathbf{q} : \{1, \ldots, n\} \times [0, 1] \to [0, 1]$, where $\mathbf{q}(i|x)$ is share of quality-*i* goods at location x's stock.

Proposition 7 (Appendix A) generalizes the threshold structure of the sorting equilibrium. Now, each sorting equilibrium features up to n thresholds, where the price schedule crosses one of the products' possible values $\{v^i\}_{i=1}^n$. With richer quality structure, markdowns and direct disposal can coexist: the seller may choose to clear some quality levels through reduced prices, while disposing of the least valuable items directly.

To illustrate, suppose there are three quality levels and the lowest has no value to consumers: $v^1 = -\kappa$. Then, sales of quality-1 do not recover any of the production costs. But clearing it through sales requires transferring more surplus to the buyer and reducing transaction prices for all other qualities. Direct disposal of quality-1 items is therefore optimal. At the same time, if v^2 is sufficiently high, then the seller prefers to clear it through sales to recover her production costs.

Proposition 10 in the Online Appendix also formulates a version of the irrelevance result for the model with multiple qualities. Fixing the disposal rate and the set of product qualities cleared through sales, the seller's problem remains one-dimensional: she is indifferent across all sorting equilibria that deliver the same consumer surplus. The role of prices remains limited even with multiple quality levels.

 $^{^{28}}$ This differs from the two-store numerical example, where time is discrete, so that there is a sizable difference between initial and post-sales quality distribution at the flagship.

6.3 Heterogeneous Consumers

This section extends the baseline model to allow for vertically differentiated consumers. Consumers differ in their valuation of high-quality goods, and the seller uses different locations to segment them. Sorting emerges not only across products but also across consumer types. Outlet locations now serve a dual role: clearing low-quality inventory and segmenting consumers with low willingness to pay.

Suppose the product has binary quality, as in Section 3, but consumers vary in their willingness to pay for high quality. A consumer of type $\theta \in \Theta = [v^l, v^h]$ values a high-quality item at θ and a low-quality item at v^l . The distribution of types admits a positive density $f(\cdot)$ over the whole support Θ , with $F(\cdot)$ denoting the cdf.

Each consumer type θ chooses a location to visit and draws a product from there. Let $\mathbf{x} : \Theta \to X$ denote the consumer strategy, mapping each type to a distinct location (\mathbf{x} is injective). Expected payoff of type- θ consumer in a market outcome ($\mathbf{p}, \mathbf{x}, \mathbf{q}$) is:

$$V^{B}(\mathbf{p}, \mathbf{x}, \mathbf{q}|\theta) = \mathbf{q}(\mathbf{x}(\theta))(\theta - \mathbf{p}(\mathbf{x}(\theta)))_{+} + (1 - \mathbf{q}(\mathbf{x}(\theta)))(v^{l} - \mathbf{p}(\mathbf{x}(\theta)))_{+},$$

The consumer strategy \mathbf{x} is optimal if no consumer type wants to deviate to another location, given prices and the quality composition. Formally, for every $\theta \in \Theta$ and every $x \in X$, the market outcome satisfies *incentive compatibility (IC)*:

$$V^{B}(\mathbf{p}, \mathbf{x}, \mathbf{q}|\theta) \ge \mathbf{q}(x)(\theta - \mathbf{p}(x))_{+} + (1 - \mathbf{q}(x))(v^{l} - \mathbf{p}(x))_{+}$$
(IC)

Sorting Equilibria. Proposition 5 shows how the threshold structure of the sorting equilibria generalizes to the model with heterogeneous consumers.

Proposition 5. For every equilibrium market outcome with positive sales, there exists a threshold outlet shopper $\hat{\theta} \in (v^l, v^h]$ such that all types in $(\hat{\theta}, v^h]$ shop at non-outlet locations; and types in $[v^l, \hat{\theta})$ shop at outlet locations. In addition, **x** is decreasing on $[\hat{\theta}, v^h]$.

Proof. See Appendix I.

Consumers self-sort across locations in a descending order. Higher types visit upstream locations with better quality and higher prices. The proof follows from the standard monotonicity arguments for IC-allocations.

Irrelevance Result. The equilibrium payoffs mirror those of the monopolistic screening model (Mussa and Rosen (1978)), but with an important twist: the seller cannot choose qualities in the menus directly, only induce them via sorting. And just as in the baseline model, sorting can be fully described by what happens at the threshold.

In particular, Lemma 14 (Appendix I) shows that for every threshold outlet shopper $\hat{\theta} \in (v^l, v^h]$, there exists a unique *induced quality allocation* $Q^{\hat{\theta}} : [\hat{\theta}, v^h] \to [0, 1]$, such that $\mathbf{q}(\mathbf{x}(\theta)) = Q^{\hat{\theta}}(\theta), \forall \theta \in [\hat{\theta}, v^h]$. This is because, fixing outlet shopper $\hat{\theta}$, the quality composition at each location depends only on the mass of consumers visiting upstream locations. By Proposition 5, for location $\mathbf{x}(\theta)$ this mass equals $1 - F(\theta)$, which is exogenously given. Seller's Problem. As before, the seller chooses a sorting equilibrium to maximize her profit

flow:

$$V^{S}(\mathbf{p}, \mathbf{x}, \mathbf{q}) = \int_{v^{l} < \mathbf{p}(\mathbf{x}(\theta)) \le \theta} \mathbf{p}(\mathbf{x}(\theta)) \mathbf{q}(\mathbf{x}(\theta)) f(\theta) d\theta + \int_{\mathbf{p}(\mathbf{x}(\theta)) \le v^{l}} \mathbf{p}(\mathbf{x}(\theta)) f(\theta) d\theta$$

Due to Proposition 5, the seller's problem reduces to segmenting consumer types between outlet and non-outlet locations by selecting $\hat{\theta}$.

Proposition 6. A sorting equilibrium $(\mathbf{p}, \mathbf{x}, \mathbf{q})$ with a threshold outlet shopper $\hat{\theta}$ is selleroptimal if and only if:

$$\hat{\theta} \in \underset{\hat{\theta} \in (v^l, v^h]}{\operatorname{argmax}} \int_{\hat{\theta}}^{v^h} Q^{\hat{\theta}}(\theta) \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) dF(\theta) + F(\hat{\theta}) v^l - Q^{\hat{\theta}}(\hat{\theta}) (\hat{\theta} - v^l) (1 - F(\hat{\theta})).$$

Proof. See Appendix I.

Consumer segmentation serves a dual role: it both determines who receives the low price and pins down the entire menu of quality-price pairs offered upstream.

6.4 Future Directions

In this section, I highlight some of the questions that fall outside the scope of this paper but offer promising avenues for future research.

Quality Depreciation. So far, I have assumed that the preferences for any particular product remain constant over time. But in real life, even popular designs lose customer appeal with time. For instance, in the apparel industry, this may happen due to the seasonality of products. Within this paper, one could accommodate time depreciation by assuming that, with some probability, a unit of unsold high-quality inventory loses its value and becomes of low quality.

With time depreciation, the irrelevance result for the continuous model no longer holds. The seller gets a new leverage for sorting products through their exogenous deterioration and must balance a new trade-off. The seller can "speed up" turnover and dampen the effect of depreciation by increasing the customer share of the earlier locations. By doing so, the seller improves the average quality composition and increases sales volume at high-priced locations but prevents the goods from getting damaged before reaching outlets.

Other Pricing Processes and Product Flows. My model greatly constrains how the seller leverages information from sales. As argued in the introduction, this assumption may be justified by the high cost of more nuanced pricing strategies for large inventory volumes. Nevertheless, the model leaves the question of optimal pricing and product reallocation open. In particular, would the seller benefit from non-linear reallocation of goods? Could the seller benefit from having two separate, independent lines of stores? Should the seller merge outlets for her different brand lines? Given the tractability model, these directions seem promising within the suggested framework.

Richer Market Structures. Theorem 1 characterizes sorting equilibria for all possible prices, not just the optimal ones. In particular, it applies to richer strategic environments where different sellers manage different prices. In future research, the model could be extended to allow for upstream and downstream sellers to explore whether inefficiency exacerbates with multiple sellers setting prices.

Frequency of Replenishment. The frequency of inventory replenishment offers another strategic tool for the seller to enhance product sorting efficiency. Exploring the impact of replenishment frequency, particularly in scenarios where stock-outs occur, could provide additional valuable insights.

7 Related Literature

This paper builds on the classic model of markdown pricing by Lazear (1986): a seller gradually lowers the price of a good with unknown consumer value, and short-lived consumers arrive gradually. In that setting, the good is produced once, and the seller can extract the entire surplus by waiting long enough. In my model, learning through the lack of sales is costly: unsold goods slow the arrival of new inventory, which is more likely to be high-value. Moreover, with many goods, consumers also make a strategic choice over which goods to inspect. The paper thus offers an equilibrium model of markdowns.

The paper also contributes to the dynamic pricing literature (*e.g.*, Gallego and Van Ryzin (1994), Den Boer (2015), Elmaghraby and Keskinocak (2003), Board and Skrzypacz (2016), Dilme and Li (2019)). In these papers, dynamic prices screen consumers. In mine, they sort products. Methodologically, I differ by focusing on a steady-state equilibrium that reduces the dynamic pricing to a static model.

Prices also serve as signals of expected quality, in the spirit of Wolinsky (1983), Bagwell and Riordan (1991), Delacroix and Shi (2013). However, the source of information is funda-

mentally different. These papers study informed sellers who use prices to communicate their private information. In contrast, my seller is uninformed, and prices become informative endogenously through the equilibrium sales process.

The sorting mechanism relates to the literature on learning from sales. Bergemann and Välimäki (1997), Bergemann and Välimäki (2000), Bergemann and Välimäki (2006), Bonatti (2011) assume a model of the seller who learns about the product by making sales, and the amount of information increases with the sales volume. Bose et al. (2006), Bose et al. (2008) study dynamic pricing models with information cascades driven by observed purchase history. In contrast, my model makes the absence of purchase a key source of information. This distinction introduces a novel trade-off between sorting precision and sales volume.

The equilibrium model is related to general equilibrium models of directed search with adverse selection (see Guerrieri, Julien, and Wright (2017) for a review). One side of the market has superior information about the match value, and the other chooses the terms of trade. The informed side of the market then sorts across the offered contract. In equilibrium, the terms of contracts get balanced against the probability of matching. In my model, consumers sort themselves and the products across different prices. The resulting trade-off in consumer search lets the seller price-discriminate the goods by quality. Lauermann and Wolinsky (2017) studies how well prices aggregate information in markets with search and adverse selection.

Inventory management under uncertain demand is studied extensively in operations and marketing literature. Some papers focus on how sellers learn and adjust production over time (see Silver, Pyke, and Thomas (2016) for a review). Others explore dynamic demand, where sales influence future outcomes directly (through contagion) or indirectly (through inference) (e.g., Hartung (1973), Petruzzi and Monahan (2003), Caro and Gallien (2007)). These models often treat demand as exogenous and abstract from consumer learning. The most closely related work is an empirical paper by Ngwe (2017), studying the joint pricing and inventory choice problem across a flagship and an outlet for consumer segmentation. Like my model, Ngwe (2017) assumes constant capacity and inventory flow from production to flagship to outlet. However, the model abstracts from consumer search and does not consider how markdowns facilitate indirect quality-based price discrimination.

Methodologically, this paper belongs to the literature on steady-state mechanism design, as in Madsen and Shmaya (2024) and Baccara, Lee, and Yariv (2020).

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Appendices

This section includes the appendices for the continuous model. It covers the versions of the model covered in Section 3 - Section 6.

Appendix A formulates the most general version of the model with homogeneous consumers. Appendix B includes all the proofs for this general model. Appendix C specializes to binary quality, and covers proofs for the benchmark model, direct disposal model, and a general shopping strategy model. Appendix E provides details and proofs for the reduced seller's problem for Section 4.3. Appendix OA2 analyzes the model with multiple qualities. Appendix I formalizes and analyzes the model with heterogeneous consumers.

A General Model

In this section, I formally describe the most general version of the continuous model, which allows buyers to have a more general shopping strategy, allows for direct disposal, and considers multiple quality tiers.

Quality Levels. The product comes in one of n possible qualities. Each consumer gets utility $v^i + \kappa$ from a product of quality i, where κ is the seller's per-unit production cost (uniform across quality types). For notational simplicity, the qualities are ordered so that: $v^1 < v^2 < \cdots < v^n$. In addition, it will be useful to define a fictitious product 0: $v^0 = -\infty$. Locations and Disposal. The seller manages a continuum of locations X = (0, 1]. 0 is the production plant, and 1 is a warehouse. Neither of these two can be visited by the consumers. The total stock of mass 1 is distributed uniformly across (0, 1]. The products from location 1 are destroyed at a constant rate $\gamma \geq 0$.

Prices and Quality Composition. The quality composition is described by $\mathbf{q} : \{1, \ldots, n\} \times X \to [0, 1]$, with $\mathbf{q}(i|x)$ denoting the share of quality-*i* goods in the total stock of location x. The quality composition at the production plant is given exogenously: $\mathbf{q}(i|0) = \pi(i)$ for some $\{\pi(i)\}_{i=1}^n$. The seller produces each quality *i* with a positive probability $\pi(i) \in (0, 1)$. $\mathbf{p} : X \to \mathbb{R}$ summarizes the price schedule for locations in X. $\mathbf{p}(x)$ is the price the seller receives conditional on purchase at location x, net of her replacement cost $\kappa > 0$. Both $\mathbf{p}(\cdot)$ and every $\mathbf{q}(i|\cdot)$ are Lebesgue-measurable.

Consumers. Consumers who visit at location x, draw a single product at random according to distribution $\{\mathbf{q}(i|x)\}_{i=1}^{n}$. If a consumer purchases the product of quality i at a price p, he gets a payoff of $v^{i} - p$. As before, the consumer buys the good whenever its value is weakly above the price.

The consumer strategy by the consumers is summarized by a cdf $D: [0,1] \rightarrow [0,1]$. To

clarify, D(x) denotes the mass of consumers drawing their good at the locations weakly below x^{29} Define $\delta : X \to [0, 1]$ to be the *size of an atom* at location x: $\delta(x) = D(x) - D(x)$. Location x is *visited* if the consumer strategy $D(\cdot)$ is strictly increasing at x^{30}

Consumer payoff at a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ is:

$$V^B(\mathbf{p}, D, \mathbf{q}, \gamma) = \int_{x \in X} \sum_{i=1}^n \mathbf{q}(i|x)(v^i - \mathbf{p}(x))_+ dD(x)$$

Sustained Quality Composition. The products move due to purchases, downstream reallocations, and disposal. For every market outcome m, the purchasing probability ρ_m : $X \to [0, 1]$ at a location x is the total probability of drawing a good with the value above the x's price:

$$\rho_m(x) = \sum_{i=1}^n \mathbb{1}\{\mathbf{p}(x) \le v^i\}\mathbf{q}(i|x),$$

For downstream reallocations, goods pass at the rate of downstream sales $S_m : [0, 1] \rightarrow [0, 1]$, that include fictitious sales from location 1 of size γ :

$$S_m(x) = \int_{y>x} \rho_m(x) dD(x) + \gamma \mathbb{1}\{x < 1\}.$$

The seller picks the goods for downstream reallocations from any location x at random. On the interval $(x_1, x_2]$, the outflow of products of quality i includes all purchases of this quality, and the products of this quality are reallocated downstream from x_2 . The inflow of quality i equals the mass of quality i products reallocated downstream from x_1 . The share of quality i stays in the stock of locations $(x_1, x_2]$ stays constant over time when its outflows and inflows are balanced:

$$\int_{y \in (x_1, x_2], p(y) \le v^i} \mathbf{q}(i|y) dD(y) + S_m(x_2) \mathbf{q}(i|x_2) = S_m(x_1) \mathbf{q}(i|x_1)$$
(4)

The quality composition **q** is sustained by prices, consumer strategy and disposal rate (\mathbf{p}, D, γ) on some subset of locations $Y \subseteq X$, if Equation (4) holds for each quality $i \in \{1, \ldots, n\}$ and each $(x_1, x_2] \subseteq Y$.

Sorting Equilibrium. A market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ is a sorting equilibrium if

²⁹Given that D is a cdf over [0, 1], I implicitly assume that it is increasing, D(0) = 0, D(1) = 1. In addition, D is continuous on the right with a limit on the left (corlol) on [0, 1]. In addition, since both the location 1 is not available for consumers, D is continuous at 1.

³⁰That is, for every $\Delta > 0$: $D(x) - D(x - \Delta) > 0$.
- (i) the quality composition \mathbf{q} is sustained by (\mathbf{p}, D, γ) ,
- (ii) each visited location x maximizes consumer expected payoff:

$$\sum_{i=1}^{n} \mathbf{q}(i|x)(v^{i} - \mathbf{p}(x))_{+} = \max_{y \in X} \sum_{i=1}^{n} \mathbf{q}(i|y)(v^{i} - \mathbf{p}(y))_{-}$$

In the remainder of the Appendix, I slightly abuse the notation and write $(\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ to denote a market outcome where the shopping strategy of a consumer admits a density σ , or write $(\mathbf{p}, D, \mathbf{q})$ to denote a market outcome where $\gamma = 0$.

Seller. The seller's profit flow from a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ is:

$$V^{S}(\mathbf{p}, D, \mathbf{q}, \gamma) = \int_{x \in (0,1)} \mathbf{p}(x) dS_{m}(x) - \gamma \kappa$$

B Proofs for the General Model

In this appendix, I analyze the general model of Appendix A.

Properties of Sustained Quality Compositions. I begin the analysis by characterizing the restrictions on the quality compositions that can be sustained in some sorting equilibrium. Lemma 3 establishes the basic continuity properties of such quality compositions.

Lemma 3. Consider a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$. \mathbf{q} is sustained by (\mathbf{p}, D, γ) only if it satisfies the following:

- (i) if $S_m(x) > 0$, then for every *i*, $\mathbf{q}(i|\cdot)$ is right-continuous at *x*, and is continuous at *x* if *D* is continuous at *x*,
- (ii) if $\mathbf{p}(x) \leq v^1$ or $\mathbf{p}(x) > v^n$, then $\mathbf{q}(i|\cdot)$ is left-continuous at x whenever $S_m(x-) > 0$,
- (iii) if $\mathbf{p}(x) > v^i$, then $S_m(\cdot)\mathbf{q}(i|\cdot)$ is continuous at x,
- (iv) if $\mathbf{p}(x) < v^i$ and D is discontinuous at x, then $\mathbf{q}(i|x) < \mathbf{q}(i|x-)$ whenever $\mathbf{q}(i|x-) > 0$ and $\rho_m(x) < 1$.

Proof. Suppose that **q** is sustained by (\mathbf{p}, D, γ) on X. Then, Equation (4) holds for every $(x_1, x_2] \subseteq (0, 1)$.

Part (i). Take an interval $(x, x + \Delta]$, then we must have:

$$-\int_{y\in(x,x+\Delta],\mathbf{p}(y)\leq v^{i}}\mathbf{q}(i|y)dD(y) + S_{m}(x)\mathbf{q}(i|x) - S_{m}(x+\Delta)\mathbf{q}(i|x+\Delta) = 0.$$

If **q** is sustained by (\mathbf{p}, D, γ) on X, the above must hold for any $x \in (0, 1)$ and $\Delta > 0$. We can take Δ to be arbitrarily small. Note that $\int_{y \in (x, x+\Delta], \mathbf{p}(y) \leq v^i} \mathbf{q}(i|y) dD(y)$ converges to 0 by the Squeeze Theorem:

$$0 = \lim_{\Delta \to 0} \int_{y \in (x, x+\Delta]} 1 dD(y) \ge \int_{y \in (x, x+\Delta], \mathbf{p}(y) \le v^i} \mathbf{q}(i|y) dD(y) \ge 0.$$

where the equality is due to D being right-continuous at every $x \in X$. Hence, **q** is sustained by (D, \mathbf{p}, γ) on X only if:

$$S_m(x)\mathbf{q}(i|x) = \lim_{\Delta \to 0} S_m(x+\Delta)\mathbf{q}(i|x+\Delta).$$

By the definition of downstream sales, S_m is right-continuous at every x. Then, to satisfy the above equation, $\mathbf{q}(i|x)$ must also be right-continuous at every location x, where $S_m(x) > 0$. Analogously, we can show that if D is left-continuous at x, then $\mathbf{q}(i|x)$ must be left-continuous if $S_m(x) > 0$.

Part (*ii*). Suppose $\mathbf{p}(x) \leq v^1$. We impose Equation (4) on the interval $(x - \Delta, x]$:

$$S_m(x-\Delta)\mathbf{q}(i|x-\Delta) - S_m(x)\mathbf{q}(i|x) = \int_{y\in(x-\Delta,x],\mathbf{p}(y)\leq v^i} \mathbf{q}(i|y)dD(y).$$

By the premise of the lemma's part, $\mathbf{p}(x) \leq v^1$, so that all goods are purchased from x: $\rho_m(x) = 1$. This lets us rewrite the above in the form:

$$S_m(x - \Delta)\mathbf{q}(i|x - \Delta) - S_m(x)\mathbf{q}(i|x) = \mathbf{q}(i|x) \left(S_m(x - \Delta) - S_m(x)\right) + \int_{y \in (x - \Delta, x), \mathbf{p}(y) \le v^i} \left(\mathbf{q}(i|y) - \mathbf{q}(i|x)\right) dD(y) - \int_{y \in (x - \Delta, x), \mathbf{p}(y) > v^1} \mathbf{q}(i|x)(1 - \rho_m(y)) dD(y).$$

which simplifies to:

$$S_m(x-\Delta)(\mathbf{q}(i|x-\Delta) - \mathbf{q}(i|x)) = \int_{y \in (x-\Delta,x), \mathbf{p}(y) \le v^i} (\mathbf{q}(i|y) - \mathbf{q}(i|x)) dD(y) - \int_{y \in (x-\Delta,x), \mathbf{p}(y) > v^1} \mathbf{q}(i|x)(1-\rho_m(y)) dD(y).$$

If **q** is sustained by (D, \mathbf{p}, γ) on X, this holds for every x and every $\Delta > 0$. As $\Delta \to 0$, the right-hand side is converging to 0, since $|\mathbf{q}(i|y) - \mathbf{q}(i|x)|$ and $\mathbf{q}(i|x)(1 - \rho_m(y))$ are at most 1.

Then, whenever **q** is sustained by (\mathbf{p}, D, γ) on X, we must have :

$$S_m(x-\Delta)(\mathbf{q}(i|x-\Delta)-\mathbf{q}(i|x)) \underset{\Delta \to 0}{\rightarrow} 0.$$

That is, unless $S_m(x-) = 0$, $\mathbf{q}(i|\cdot)$ is left-continuous at x. The proof for the case $\mathbf{p}(x) > v^n$ is analogous.

Part (*iii*). Suppose that $\mathbf{p}(x) > v^i$: quality *i* is not purchased at *x*. Then, Equation (4) on $(x - \Delta, x]$ is equivalent to:

$$= \int_{\mathbf{p}(x) \le v^i, y \in (x - \Delta, x)} \mathbf{q}(i|y) dD(y).$$

Taking arbitrarily small Δ , the right-hand side converges to 0. Then, we have:

$$\lim_{\Delta \to 0} S_m(x - \Delta) \mathbf{q}(i|x - \Delta) - S_m(x) \mathbf{q}(i|x) = 0.$$

That is, $S_m(\cdot)\mathbf{q}(i|\cdot)$ is left-continuous at x. $S_m(\cdot)$ is right-continuous, and $\mathbf{q}(i|\cdot)$ is rightcontinuous by part (i) whenever $S_m(x) > 0$. Together, these deliver continuity of $S_m(\cdot)\mathbf{q}(i|\cdot)$ at x.

Part (*iv*). Suppose that $\mathbf{p}(x) < v^i$: quality *i* is purchased at *x*. Equation (4) is satisfied for the interval $(x - \Delta, x]$ whenever:

$$S_m(x - \Delta)\mathbf{q}(i|x - \Delta) - S_m(x)\mathbf{q}(i|x) = \int_{\mathbf{p}(x) \le v^i, y \in (x - \Delta, x]} \mathbf{q}(i|y) dD(y)$$
$$= \delta(x)\mathbf{q}(i|x) + \int_{\mathbf{p}(x) \le v^i, y \in (x - \Delta, x)} \mathbf{q}(i|y) dD(y) dy$$

Taking the limit of both sides as $\Delta \to 0$, we obtain:

$$S_m(x-)\mathbf{q}(i|x-) - S_m(x)\mathbf{q}(i|x) = \delta(x)\mathbf{q}(i|x)$$

As $\mathbf{p}(x) \leq v^i$, then $S_m(x-) = \delta(x)\rho_m(x) + S_m(x)$ and the above implies:

$$\mathbf{q}(i|x) = \mathbf{q}(i|x-)\frac{\delta(x)\rho_m(x) + S_m(x)}{\delta(x) + S_m(x)}$$

 $\frac{\delta(x)\rho_m(x)+S_m(x)}{\delta(x)+S_m(x)}$ is strictly lower than 1 whenever $\mathbf{q}(i|x-) > 0$ and $\rho_m(x) < 1$.

Lemma 4 summarizes the key sorting restrictions. If a quality is not purchased over a certain interval, its mass in downstream reallocations remains constant during that interval.

In addition, there is only some sorting for any two qualities if consumer purchasing decisions are different across these qualities.

Lemma 4. Consider a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$. Suppose that $\mathbf{p}(x) \in (v^i, v^{i+1})$ D-a.s. on $[x_1, x_2]$, and $S_m(x_2) > 0$. Then, **q** is sustained by (\mathbf{p}, D, γ) on $[x_1, x_2]$ only if $S_m(\cdot)\mathbf{q}(l|\cdot)$ is constant over $[x_1, x_2]$ for every $l \leq i$. In addition, whenever either D is continuous on $[x_1, x_2]$, or i = 0, **q** is sustained by (\mathbf{p}, D, γ) on $[x_1, x_2]$ if and only if:

- (i) for every $l \leq i$, $S_m(\cdot)\mathbf{q}(l|\cdot)$ is constant over $[x_1, x_2]$,
- (ii) for every l > i, $\frac{\mathbf{q}(l)}{\sum_{k>i} \mathbf{q}(k|\cdot)}$ is constant over $[x_1, x_2]^{.31}$

Proof. Note that **q** is sustained by (\mathbf{p}, D, γ) over $[x_1, x_2]$ whenever for any $x \in (x_1, x_2]$, any $(x - \Delta, x] \subseteq [x_1, x_2]$ and $\forall l \in \{1, \dots, n\}$:

$$\int_{y \in (x-\Delta,x], \mathbf{p}(y) \le v^l} \mathbf{q}(l|y) dD(y) = S_m(x-\Delta)\mathbf{q}(l|x-\Delta) - S_m(x)\mathbf{q}(l|x).$$
(5)

Only if and part (i). Take any quality type $l \leq i$, which is not purchased over $[x_1, x_2]$. Then, $\int_{y \in (x-\Delta,x], \mathbf{p}(y) \le v^l} \mathbf{q}(l|y) dD(y) = 0$, and Equation (5) holds if and only if:

$$S_m(x - \Delta)\mathbf{q}(l|x - \Delta) = S_m(x)\mathbf{q}(l|x)$$

As the above must hold for every $(x - \Delta, x] \subseteq [x_1, x_2]$, this is equivalent to $S_m(\cdot)\mathbf{q}(l|\cdot)$ being constant over $|x_1, x_2|$.

Part (*ii*). First, let me show the only if direction. Fix some product quality l > i (purchased a.e. on $[x_1, x_2]$) and assume by way of contradiction that there exists a pair of locations $\tilde{x}_1 < \tilde{x}_2$, where $\frac{\mathbf{q}(l|\tilde{x}_1)}{\sum_{k>i} \mathbf{q}(k|\tilde{x}_1)} > \frac{\mathbf{q}(l|\tilde{x}_2)}{\sum_{k>i} \mathbf{q}(k|\tilde{x}_2)}$ (the other case is symmetric).

By Lemma 3, $\mathbf{q}(k|\cdot)$ is continuous for every $k \in \{1, \ldots, n\}$ on $[x_1, x_2]$ under the premise of the additional part of the lemma. Then, there exists some $\tilde{y}_1 < \tilde{x}_2$, such that for all $y \in (\tilde{y}_1, \tilde{x}_2], \frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k>i} \mathbf{q}(k|\tilde{y}_1)} > \frac{\mathbf{q}(l|y)}{\sum_{k>i} \mathbf{q}(k|y)}.^{32} \mathbf{q} \text{ is sustained by } (D, \mathbf{p}, \gamma) \text{ over } (\tilde{y}_1, \tilde{x}_2] \text{ only if:}$

$$0 = -\int_{y \in (\tilde{y}_1, \tilde{x}_2], \mathbf{p}(y) \le v^l} \mathbf{q}(l|y) dD(y) + S_m(\tilde{y}_1) \mathbf{q}(l|\tilde{y}_1) - S_m(\tilde{x}_2) \mathbf{q}(l|\tilde{x}_2)$$

By the premise of the lemma, $\mathbf{p}(\cdot) \in (v^i, v^{i+1})$ (D-a.s.) on $[x_1, x_2]$: (D-a.s.), the purchasing probability is $\rho_m(\cdot) = \sum_{k>i} \mathbf{q}(k|\cdot)$, the probability that the quality is above *i*. We can rewrite

³¹With a convention that $\frac{\mathbf{q}(j|x)}{\sum_{k>i}\mathbf{q}(k|x)} = 1$ when $\sum_{k>i}\mathbf{q}(k|x) = 0$. ³²For instance, we may take $\tilde{y}_1 = \sup\left\{y : \frac{\mathbf{q}(l|y)}{\sum_{k>i}\mathbf{q}(k|y)} = \frac{\mathbf{q}(l|\tilde{x}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_1)}\right\}$.

the above equation as:

$$0 = -\int_{y \in (\tilde{y}_{1}, \tilde{x}_{2}], \mathbf{p}(y) \in (v^{i}, v^{i+1})} \frac{\mathbf{q}(l|y)}{\sum_{k>i} \mathbf{q}(k|y)} \rho_{m}(y) dD(y) + S_{m}(\tilde{y}_{1}) \sum_{k>i} \mathbf{q}(k|\tilde{y}_{1}) \frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i} \mathbf{q}(k|\tilde{y}_{1})} - S_{m}(\tilde{x}_{2}) \sum_{k>i} \mathbf{q}(k|\tilde{x}_{2}) \frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i} \mathbf{q}(k|\tilde{x}_{2})}$$

By our premise $\frac{\mathbf{q}(l|y)}{\sum_{k>i}\mathbf{q}(k|y)} < \frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)}$ for all $y \in (\tilde{y}_1, \tilde{x}_2]$, then we have:

$$\begin{split} 0 &> -\frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})} \left(S_{m}(\tilde{y}_{1}) - S_{m}(\tilde{x}_{2})\right) + S_{m}(\tilde{y}_{1}) \sum_{k>i}\mathbf{q}(k|\tilde{y}_{1}) \frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})} \\ &- S_{m}(\tilde{x}_{2}) \sum_{k>i}\mathbf{q}(k|\tilde{x}_{2}) \frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_{2})} \\ &= -\frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})} \sum_{k\leq i}\mathbf{q}(k|\tilde{y}_{1})S_{m}(\tilde{y}_{1}) + \frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_{2})} \sum_{k\leq i}\mathbf{q}(k|\tilde{x}_{2})S_{m}(\tilde{x}_{2}) \\ &+ S_{m}(\tilde{x}_{2}) \left(\frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})} - \frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_{2})}\right). \end{split}$$

From part (i), we can replace $\sum_{k \leq i} \mathbf{q}(k|\tilde{y}_1) S_m(\tilde{y}_1)$ with $\sum_{k \leq i} \mathbf{q}(k|\tilde{x}_2) S_m(\tilde{x}_2)$ to get:

$$0 > \left(\frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_{2})} - \frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})}\right) \sum_{k\leq i} \mathbf{q}(k|\tilde{x}_{2})S_{m}(\tilde{x}_{2}) + S_{m}(\tilde{x}_{2}) \left(\frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})} - \frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_{2})}\right) \\ = S_{m}(\tilde{x}_{2}) \left(1 - \sum_{k\leq i}\mathbf{q}(k|\tilde{x}_{2})\right) \left(\frac{\mathbf{q}(l|\tilde{y}_{1})}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_{1})} - \frac{\mathbf{q}(l|\tilde{x}_{2})}{\sum_{k>i}\mathbf{q}(k|\tilde{x}_{2})}\right).$$

Finally, by our premise, $\frac{\mathbf{q}(l|\tilde{y}_1)}{\sum_{k>i}\mathbf{q}(k|\tilde{y}_1)} > \frac{\mathbf{q}(l|\tilde{x}_2)}{\sum_{k>i}\mathbf{q}(k|y)}$. We obtain a contradiction: 0 > 0.

To prove the if direction, suppose that $\frac{\mathbf{q}(l|y)}{\sum_{k>i} \mathbf{q}(k|y)}$ remains constant over $[x_1, x_2]$. We now check that Equation (5) is satisfied for every $(x - \Delta, x] \subseteq [x_1, x_2]$:

$$\begin{split} &\int_{y\in(x-\Delta,x],\mathbf{p}(y)\in(v^{i},v^{i+1})} \frac{\mathbf{q}(l|y)}{\sum_{k>i} \mathbf{q}(k|y)} \rho_{m}(y) dD(y) + S_{m}(x-\Delta) \mathbf{q}(l|x-\Delta) - S_{m}(x) \mathbf{q}(l|x) \\ &= -\frac{\mathbf{q}(l|x)}{\sum_{k>i} \mathbf{q}(k|x)} (S_{m}(x-\Delta) - S_{m}(x)) + S_{m}(x-\Delta) \sum_{k>i} \mathbf{q}(k|x-\Delta) \frac{\mathbf{q}(l|x)}{\sum_{k>i} \mathbf{q}(k|x)} \\ &- S_{m}(x) \sum_{k>i} \mathbf{q}(k|x) \frac{\mathbf{q}(l|x)}{\sum_{k>i} \mathbf{q}(k|x)} \\ &= \frac{\mathbf{q}(l|x)}{\sum_{k>i} \mathbf{q}(k|x)} \left[S_{m}(x) \sum_{k\leq i} \mathbf{q}(k|x) - S_{m}(x-\Delta) \sum_{k\leq i} \mathbf{q}(k|x-\Delta) \right]. \end{split}$$

By the lemma part (i), $S_m(x) \sum_{k>i} \mathbf{q}(k|x) - S_m(x-\Delta) \sum_{k>i} \mathbf{q}(k|x-\Delta)$, and hence Equation (5) is satisfied.

Sales Collapse. Lemma 5 formalizes the general result: if quality i is never purchased or never destroyed, the sales collapse in the sorting equilibrium. The intuition is the same as in the binary quality. When not cleared from the stock, any quality crowds out all higher-quality items.

Lemma 5. Consider a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$, such that $\mathbf{p}(x) > v^i$ for all $x < \hat{x}$ (D-a.s.) and $S_m(\hat{x}) = 0$. If m is a sorting equilibrium, then the sales collapse: $S_m(0) = 0$.

Proof. Suppose the statement of the lemma is not true. $S_m(\hat{x}-) = 0$, but the total sales are positive: $S_m(0) > 0$. Then, by Lemma 4 part (i), $S_m(x)\mathbf{q}(i|x)$ is constant on $(0, \hat{x})$ and is right-continuous at 0 by Lemma 3 (i) and right-continuity of $S_m(\cdot)$.

Then, we must have:

$$S_m(0)\pi(i) = \lim_{x \to 0} S_m(x)\mathbf{q}(i|x) = S_m(\hat{x}-)\mathbf{q}(i|\hat{x}-) = 0.$$

By assumption, $\pi(i) \in (0, 1)$, and we obtain a contradiction.

Imperfect Sorting. Lemma 6 states that the seller can never perfectly discover the quality type in a sorting equilibrium with positive sales. She always makes some mistakes when reallocating the goods downstream.

Lemma 6. Consider a market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ with $\mathbf{p}(x) > v^i$ for all $x < \hat{x}$. If m is a sorting equilibrium with positive sales, then $\sum_{k < i} \mathbf{q}(k|\hat{x}-) < 1$.

Proof. Suppose the statement of the lemma is not true. First, assume that there exists some upstream location $x < \hat{x}$, where the discovery of lower qualities is perfect: $\sum_{k \le i} \mathbf{q}(k|x) = 1$. Let \tilde{x} be the first such location:

$$\tilde{x} = \inf\{x < \hat{x} : \sum_{k \le i} \mathbf{q}(k|x) = 1\}.$$

Step 1.1: $\sum_{k\leq i} \mathbf{q}(k|x) = 1$ for all $x \in [\tilde{x}, \hat{x})$. By Lemma 4, \mathbf{q} is sustained by (\mathbf{p}, D, γ) only if $\sum_{k\leq i} \mathbf{q}(k|x)S_m(x)$ remains constant over $[0, \hat{x})$. By the premise of the lemma, total sales are positive, $S_m(0) > 0$. Then, by Lemma 5, \hat{x} has positive downstream sales: $S_m(\hat{x}-) > 0$. As downstream sales are non-increasing, and $\sum_{k\leq i} \mathbf{q}(k|x)S_m(x)$ remains constant, $\sum_{k\leq i} \mathbf{q}(k|\cdot)$ is non-decreasing on $[0, \hat{x})$. Then, $\sum_{k\leq i} \mathbf{q}(k|x) = 1$ on (\tilde{x}, \hat{x}) . In addition, as $S_m(\hat{x}-) > 0$, by

Lemma 3 part (i), **q** is right-continuous on $[0, \hat{x})$. Then, by the definition of \tilde{x} , $\sum_{k \leq i} \mathbf{q}(k|\tilde{x}) = 1$.

Step 1.2: $\sum_{k\leq i} \mathbf{q}(k|\cdot)$ is continuous at \tilde{x} . Suppose not. By Lemma 3 part (i), D admits an atom at \tilde{x} . If $\sum_{k\leq i} \mathbf{q}(k|x)$ is discontinuous at \tilde{x} , then it must have an upward jump at \tilde{x} : $\sum_{k< i} \mathbf{q}(k|\tilde{x}-) < 1$ (as $\sum_{k\leq i} \mathbf{q}(k|\cdot)$ is non-decreasing on $[0, \hat{x})$ from the proof of Step 1.1). But if $\sum_{k\leq i} \mathbf{q}(k|\tilde{x}) = 1$ and $\mathbf{p}(\tilde{x}) > v^i$, location \tilde{x} makes no sales. In this case, $S_m(\cdot)$ is continuous at \tilde{x} . By Lemma 3 (*iii*), $S_m(\cdot) \sum_{k\leq i} \mathbf{q}(k|\cdot)$ is continuous at \tilde{x} . We get a contradiction: $\sum_{k\leq i} \mathbf{q}(k|x)S_m(x)$ makes an upward jump at \tilde{x} , where it must remain constant.

Step 1.3. Using $\sum_{k \leq i} \mathbf{q}(k|x) S_m(x)$ is constant over $[0, \hat{x}]$ and $\sum_{k \leq i} \mathbf{q}(k|\tilde{x}) = 1$:

$$\sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta) S_m(\tilde{x} - \Delta) = \sum_{k \le i} \mathbf{q}(k|\tilde{x}) S_m(\tilde{x}) = S_m(\tilde{x})$$

for any $\tilde{x} > \Delta > 0$. Then, for any such Δ :

$$\frac{1}{\sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta)} = \frac{S_m(\tilde{x} - \Delta)}{S_m(\tilde{x})} = 1 + \frac{\int_{y \in (\tilde{x} - \Delta, \tilde{x}]} \rho_m(y) dD(y)}{S_m(\tilde{x})}$$

As $\mathbf{p}(\cdot) > v^i$ on $(0, \hat{x})$, then at most qualities strictly above *i* are purchased on $[0, \tilde{x}]$: $\rho_m(y) \le 1 - \sum_{k \le i} \mathbf{q}(k|y)$ for all $y \in (0, \tilde{x}]$. And because $\sum_{k \le i} \mathbf{q}(k|y)$ is non-decreasing, then $\rho_m(y) \le 1 - \sum_{k \le i} \mathbf{q}(k|\tilde{x})$ for all $y \in [\tilde{x} - \Delta, \tilde{x}]$. Then, we have:

$$\frac{1}{\sum_{k\leq i}\mathbf{q}(k|\tilde{x}-\Delta)} \leq 1 + (1 - \sum_{k\leq i}\mathbf{q}(k|\tilde{x}-\Delta))\frac{\int_{y\in(\tilde{x}-\Delta,\tilde{x})}dD(y)}{S_m(\tilde{x})}.$$

Rearranging, we get:

$$\frac{1 - \sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta)}{\sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta)} \le \frac{\left(1 - \sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta)\right) \int_{y \in (\tilde{x} - \Delta, \hat{x})} dD(y)}{S_m(\tilde{x})}$$

By the definition of \tilde{x} , $\sum_{k \leq i} \mathbf{q}(k|\tilde{x} - \Delta) < 1$ for every $\Delta > 0$, hence the above inequality is only satisfied when:

$$\frac{1}{\sum_{k \le i} \mathbf{q}(k|\tilde{x} - \Delta)} \le \frac{\int_{y \in (\tilde{x} - \Delta, \tilde{x})} dD(y)}{S_m(\tilde{x})}.$$

Taking the limit as $\Delta \to 0$, the right-hand side is converging to 0. If the premise is true, the left-hand side must converge to 1 by continuity of $\sum_{k < i} \mathbf{q}(k|x)$ at \tilde{x} from Step 1.2. We get a

contradiction. The proof is analogous for the case where $\sum_{k \leq i} \mathbf{q}(k|x) < 1$ for all $x < \hat{x}$ but converges to 1.

Threshold Structure. Proposition 7 establishes the threshold structure for all sorting equilibria in the general model of Appendix A.

Proposition 7. Consider a sorting equilibrium $(\mathbf{p}, D, \mathbf{q}, \gamma) \in \mathcal{E}$ with positive sales. Let $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \leq v^i\}$, with a convention that $\hat{x}_i = 1$ whenever $\mathbf{p}(x) > v^i$ for all $x \in X$.

- (i) \hat{x}_i is decreasing in *i*,
- (*ii*) $\mathbf{p}(\cdot) \in (v^i, v^{i+1})$ on $[\hat{x}_{i+1}, \hat{x}_i]$ (D-a.s.),
- (iii) if the price is ever above v^1 at visited locations, i.e., $\int_{y \in (0,\hat{x}_1)} dD(y) > 0$, then it is above v^1 at all visited locations, i.e., $\int_{\{y:\mathbf{p}(y) \ge v^1\}} dD(y) = 1$.

Proof. **Part** (i). The first part is straightforward, as we take infimum over a larger set as i increases: $\{x \in X : \mathbf{p}(x) \le v^i\} \subseteq \{x \in X : \mathbf{p}(x) \le v^j\}$ for any j > i.

Part (*ii*). The statement is trivially true whenever $\mathbf{p}(\cdot) > v^n$ on X (in this case, $\hat{x}_i = 1, \forall i$). **Step 1**: *if* $\mathbf{p}(x) \leq v^i$ and $\sum_{k>i} \mathbf{q}(j|x) > 0$, then $\mathbf{p}(\cdot) \leq v^i$ D-a.s. on [x, 1). Suppose the statement is false. Let \tilde{x}_i denote the location where the statement is false for "the first time":

$$\tilde{x}_i = \sup\left\{ y > x : \int_{z \in (x,y), \mathbf{p}(z) > v^i} dD(z) = 0 \right\}$$

Case 1.1.: there is an atom at \tilde{x}_i , $\delta(\tilde{x}_i) > 0$. Then, by the definition of \tilde{x}_i : $\mathbf{p}(\tilde{x}_i) > v^i$. That is, at \tilde{x}_i , we have an upward jump in price. If $\rho_m(\tilde{x}_i) = 0$, then the consumer receives a zero payoff at \tilde{x}_i . By the premise, $\sum_{k>i} \mathbf{q}(j|x) > 0$, and the consumer may get a strictly positive payoff at location x. Alternatively, if $\rho_m(\tilde{x}_i) > 0$, the consumer can achieve a strictly higher payoff at the locations that are in the left neighborhood of \tilde{x}_i : they have discretely better price and offer the quality composition that is at least as attractive as \tilde{x}_i (by Lemma 3, part (*iv*)). Again, we obtain a contradiction.

Case 1.2.: no atom at \tilde{x}_i , $\delta(\tilde{x}_i) = 0$. Given the definition of \tilde{x}_i , for any $\Delta > 0$:

$$\int_{z \in ([\tilde{x}_i, \tilde{x}_i + \Delta], \mathbf{p}(z) > v^i} dD(z) > 0.$$

There exists Δ small enough so that $\mathbf{p}(\cdot) \in (v^i, v^{i+1})$ over $[\tilde{x}_i, \tilde{x}_i + \Delta]$, *D*-a.s.. Indeed, by Lusin's Theorem, if $\mathbf{p}(\cdot)$ is Lebesgue-measurable, it coincides with a continuous function except possibly for a zero-measure set. By assumption, $\mathbf{p}(\cdot) > v^i$ on a positive measure of $[\tilde{x}_i, \tilde{x}_i + \Delta]$, then it must be that $\mathbf{p}(\cdot) > v^i$ a.e. on $[\tilde{x}_i, \tilde{x}_i + \Delta]$ for some Δ . In addition, for small enough Δ , consumer shops with zero probability at the locations where $\mathbf{p}(\cdot) > v^{i+1}$ as $\mathbf{q}(l|\cdot)$ is continuous for every l at \tilde{x}_i by Lemma 3 (i), and $\mathbf{p}(\tilde{x}_i-) \leq v^i$.

Then, for small enough Δ , $\mathbf{p}(y) \in (v^i, v^{i+1})$ on $[\tilde{x}_i, \tilde{x}_i + \Delta]$ *D*-a.s., and *D* is continuous on $[\tilde{x}_i, \tilde{x}_i + \Delta]$ (as *D* is continuous at \tilde{x}_i by our premise and is right-continuous everywhere). Consumer shops with a positive probability at the locations inside $[\tilde{x}_i, \tilde{x}_i + \Delta]$ where he gets a payoff:

$$\sum_{k=1}^{n} \mathbf{q}(k|x)(v^{k} - \mathbf{p}(x))_{+} < \sum_{k>i} \mathbf{q}(k|x)(v^{k} - v^{i}) \le \sum_{k>i} \mathbf{q}(k|\tilde{x}_{i})(v^{k} - v^{i}).$$

where the first inequality holds due to $\mathbf{p}(y) \in (v^i, v^{i+1})$ on $[\tilde{x}_i, \tilde{x}_i + \Delta]$ *D*-a.s.; and the second inequality holds because the quality composition of $\mathbf{q}(k|x)$ is non-increasing over $[\tilde{x}_i, \tilde{x}_i + \Delta]$ for every k > i by Lemma 4 parts (*i*) and (*ii*). Again, we obtain a contradiction with the optimality of the consumer's strategy: there is a profitable deviation towards \tilde{x}_i .

Step 2. By the definition of \hat{x}_i , either $\mathbf{p}(\hat{x}_i) \leq v^i$, or for any $\Delta > 0$, there exists $x \in (\hat{x}_i, \hat{x}_i + \Delta)$ such that $\mathbf{p}(x) \leq v^i$. By Lemma 6, $\sum_{k>i} \mathbf{q}(j|\hat{x}_i) > 0$, and by Lemma 3 (i), the same inequality is preserved in the right neighborhood of \hat{x}_i . Either way, we get that the premise of Step 1 is satisfied either at \hat{x}_i or in the right neighborhood of \hat{x}_i . Part (*ii*) of the lemma follows.

Part (*iii*). Suppose not and let:

$$\hat{y}_1 = \inf\{x \in X : \mathbf{p}(x) < v^1\},\\ \tilde{y}_1 = \inf\{y \le \hat{x}_1 : \int_{z \in (y, \hat{x}_1)} dD(y) = 0\}.$$

Case 1: $\hat{x}_1 = \hat{y}_1$. Let $\tilde{y}_1 = \inf \left\{ y \leq \hat{x}_1 : \int_{z \in (y, \hat{x}_1)} dD(y) = 0 \right\}$. By Lemma 4 part (*ii*), $\mathbf{q}(l|\cdot)$ remains constant over $(\tilde{y}_1, \hat{x}_1]$ for every *l*. There can be no atom at \tilde{y}_1 , as at \tilde{y}_1 , consumer's payoff is lower than that at \hat{x}_1 :

$$\sum_{k=1}^{n} \mathbf{q}(k|\tilde{y}_1)(v^k - \mathbf{p}(\tilde{y}_1))_+ < \sum_{k=1}^{n} \mathbf{q}(k|\tilde{y}_1)(v^k - \mathbf{p}(\hat{x}_1)) = \sum_{k=1}^{n} \mathbf{q}(k|\hat{x}_1)(v^k - \mathbf{p}(\hat{x}_n))$$

where we used (by the definition of \hat{y}_1): $\mathbf{p}(\tilde{y}_1) \ge v_1 > \mathbf{p}(\hat{y}_1) = \mathbf{p}(\hat{x}_1)$.

Since there is no atom at \tilde{y}_1 , Lemma 3 delivers $\mathbf{q}(l|\cdot)$ is continuous at \tilde{y}^1 for every l. But then, the consumer must shop with zero probability in a left neighborhood of \tilde{y}^1 , as all these locations hold only marginally different quality composition but a discretely higher price compared to \hat{x}_1 . But this is only possible if $\tilde{y}^1 = 0$, which contradicts our assumption $\int_{(0,\hat{x}_1)} dD(y) > 0.$

Case 2: $\hat{y}_1 > \hat{x}_1$. If $S_m(\hat{y}_1-) = 0$, the statement would be true. Conversely, suppose $S_m(\hat{y}_1-) > 0$. By Lemma 3 part (*ii*) $\mathbf{q}(i|\cdot)$ is continuous at \hat{y}_1 for every *i*. By Lemma 4 (*ii*), $\mathbf{q}(i|\hat{y}_1-) = \mathbf{q}(i|\hat{y}_1) = \mathbf{q}(i|\hat{x}_1)$ for every *i*. Again, we obtain a contradiction since either $\mathbf{q}(\cdot)$ remains constant over (\tilde{y}_1, \hat{y}_1) , and the consumers suboptimally shop at high-priced locations; or consumers suboptimally shop at some outlet locations in (\hat{x}_1, \hat{y}_1) by paying a higher price for the same quality composition as at \hat{y}_1 .

C Binary Quality

In this section, I provide results for a special case of a general model (see Appendix A) with a binary quality type.

Lemma 7. Consider a \hat{x} -threshold market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma) \in \mathcal{E}$. If m is a sorting equilibrium with positive sales, then the total surplus is given by:

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) = \left(\int_{\mathbf{p}(x) \le v^l} dD(x) + \gamma\right) \frac{1 - \mathbf{q}(\hat{x})}{1 - \pi} \left(\pi v^h + (1 - \pi)v^l\right) - \gamma \mathbf{q}(\hat{x})(v^h - v^l) - \gamma(\kappa + v^l)$$

Proof. Note that if \hat{x} is an outlet threshold, then either D is continuous at \hat{x} or $\mathbf{p}(\hat{x}) \leq v^{l}$. In either case, \mathbf{q} is continuous at \hat{x} by Lemma 3. By Lemma 4 (*ii*), the quality composition remains constant over $[\hat{x}, 1)$ and coincides with $\mathbf{q}(\hat{x}) = \mathbf{q}(\hat{x}-)$ (*D*-a.s.). Then, the total surplus is given by:

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) = \int_{x \in (0, \hat{x})} v^h \mathbf{q}(x) dD(x) + \int_{\mathbf{p}(x) \le v^l} dD(x) \left[\mathbf{q}(\hat{x}) v^h + (1 - \mathbf{q}(\hat{x})) v^l \right] - \gamma \kappa$$
$$= v^h \left(S_m(0) - S_m(\hat{x} -) \right) + \int_{\mathbf{p}(x) \le v^l} dD(x) \left[\mathbf{q}(\hat{x}) v^h + (1 - \mathbf{q}(\hat{x})) v^l \right] - \gamma \kappa$$

Recall that $S_m(x)(1-\mathbf{q}(x))$ is constant on $(0, \hat{x})$ by Lemma 4, and $S_m(\hat{x}-) = \int_{\mathbf{p}(x) \leq v^l} dD(x) + \gamma$. Hence, we obtain:

$$TS(\mathbf{p}, D, \mathbf{q}, \gamma) = \left(\int_{\mathbf{p}(x) \le v^l} dD(x) + \gamma \right) v^h \left(\frac{1 - \mathbf{q}(\hat{x})}{1 - \pi} - 1 \right) \\ + \int_{\mathbf{p}(x) \le v^l} dD(x) \left[\mathbf{q}(\hat{x}) v^h + (1 - \mathbf{q}(\hat{x})) v^l \right] - \gamma \kappa$$

$$= \left(\int_{\mathbf{p}(x) \le v^l} dD(x) + \gamma \right) (1 - \mathbf{q}(\hat{x})) \left(\frac{\pi}{1 - \pi} v^h + v^l \right) - \gamma \mathbf{q}(\hat{x}) (v^h - v^l) - \gamma (\kappa + v^l)$$

Shape of Equilibrium Quality Composition. For threshold market outcomes, we can characterize the sustained quality composition using Lemma 1. In particular, on the interval containing non-outlet locations $(0, \hat{x})$, the evolution of the relative likelihood between the two quality levels is captured by the Lambert function $W : \mathbb{R}_{++} \to \mathbb{R}_{+}$, where W(x) is implicitly defined as:

$$W(x)e^{W(x)} = x$$

Lemma 8. Consider \hat{x} -threshold market outcome $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma)$, with a consumer strategy admitting a density σ . Suppose total sales are positive, $S_m(0) > 0$, then \mathbf{q} is sustained by (\mathbf{p}, σ) on $[0, \hat{x}]$ if and only if for every $x \in [0, \hat{x}]$:

$$\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} = W\left(\frac{\pi}{1-\pi} \exp\left[\frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y) dy}{(1-\mathbf{q}(\hat{x}))S_m(\hat{x})}\right]\right)$$

Proof. By Lemma 4, \mathbf{q} is sustained by (\mathbf{p}, σ) on $[0, \hat{x}]$ if and only if $S_m(x)(1 - \mathbf{q}(x))$ remains constant over $[0, \hat{x}]$. If consumer strategy admits a density, then S_m is differentiable almost everywhere on X. In particular, for almost every $x \in [0, \hat{x}]$:

$$\partial_x S_m(x) = -\sigma(x)\mathbf{q}(x)$$

Hence, **q** is also almost-everywhere differentiable on $[0, \hat{x}]$, with a derivative:

$$\partial_x \mathbf{q}(x) = -\mathbf{q}(x)(1 - \mathbf{q}(x))\frac{\sigma(x)}{S_m(x)}$$
$$= -\mathbf{q}(x)(1 - \mathbf{q}(x))^2 \frac{\sigma(x)}{(1 - \mathbf{q}(\hat{x}))S_m(\hat{x})}$$

Then, from the above, we can solve out for the cumulative number of shoppers at all locations below x for any $x < \hat{x}$ such that $\mathbf{q}(x) > 0$ —which holds everywhere on $[0, \hat{x}]$ by Lemma 6. We obtain that

$$\frac{\int_0^x \sigma(y) dy}{(1 - \mathbf{q}(\hat{x})) S_m(\hat{x})} = \int_0^x -\frac{\partial_y \mathbf{q}(y)}{\mathbf{q}(y)(1 - \mathbf{q}(y))^2} dy$$

$$= \int_{\mathbf{q}(x)}^{\pi} \frac{1}{q(1-q)^2} dq = \ln\left(\frac{\pi}{1-\pi} \frac{1-\mathbf{q}(x)}{\mathbf{q}(x)}\right) + \frac{\pi}{1-\pi} - \frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}$$

Rearranging, we get:

$$\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} + \ln\left(\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}\right) = \ln\left(\frac{\pi}{1-\pi}\right) + \frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{(1-\mathbf{q}(\hat{x}))S_m(\hat{x})}$$
$$\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} \exp\left[\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}\right] = \frac{\pi}{1-\pi} \exp\left[\frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{(1-\mathbf{q}(\hat{x}))S_m(\hat{x})}\right]$$
$$\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} = W\left(\frac{\pi}{1-\pi} \exp\left[\frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{(1-\mathbf{q}(\hat{x}))S_m(\hat{x})}\right]\right)$$

Convergence to Time-Invariant Solution: Simulations. I check if the long-run interpretation of the sustained quality composition is consistent with the simulations.

Figure 7 plots the evolution of the quality composition. I discretize time, setting the length of one period (the mass of consumers within a period) to 0.001. \mathbf{q}_t denotes the quality composition at period t. I assume that at period 0, the quality composition at all locations is the same and coincides with the production plant π . \mathbf{q} denotes the sustained quality composition. Simulations depicted in Figure 7 confirm convergence to the steady-state quality composition \mathbf{q} .



Figure 7: Evolution of the Quality Composition

Note: the figure plots the quality composition after 100, 10^3 and 10^6 periods when 0.5 is the outlet threshold, and the consumer strategy of consumers attention uniformly across all locations at all periods.

Bounds of Sorting. Here, I derive the bounds on sorting under a general consumer strategy. Define two boundary functions $\lambda : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $\Lambda : [0, 1] \times [0, 1] \rightarrow [0, 1]$:

$$1 + \gamma = (D_o + \gamma)(1 - \lambda(D_o, \gamma)) \left[\ln\left(\frac{\pi}{1 - \pi} \frac{1 - \lambda(D_o, \gamma)}{\lambda(D_o, \gamma)}\right) + \frac{1}{1 - \pi} \right]$$

and

$$\Lambda(D_o, \gamma) = \pi \frac{D_o + \gamma}{D_o + \gamma + (1 - D_o)(1 - \pi)}$$

Lemma 9. Consider a \hat{x} -threshold market outcome $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$ with positive total sales, $S_m(0) > 0$. Suppose that consumer strategy D is discontinuous at finitely many points. If m is a sorting equilibrium, then $\mathbf{q}(\hat{x}) \in \left[\lambda\left(\int_{y \in [\hat{x}, 1]} dD(y)\right), \Lambda\left(\int_{y \in [\hat{x}, 1]} dD(y)\right)\right]$

(i) If D admits no atoms at non-outlet locations, then $\mathbf{q}(\hat{x}) = \lambda \left(\int_{y \in [\hat{x},1]} dD(y) \right)$.

(ii) If there is a unique visited non-outlet location, then $\mathbf{q}(\hat{x}) = \Lambda\left(\int_{y \in [\hat{x},1]} dD(y), \gamma\right)$.

(iii) If D admits finitely many atoms on $(0, \hat{x})$, then $\mathbf{q}(\hat{x}) > \lambda \left(\int_{y \in [\hat{x}, 1]} dD(y), \gamma \right)$. Proof. For brevity, let me denote $\int_{y \in [\hat{x}, 1]} dD(y)$ with D_o .

Step 1: only outlets. First, note that the statement is true for $D_o = 1$, since all consumers shop at the outlet locations. If **q** is sustained by (\mathbf{p}, D, γ) , then $\mathbf{q}(\hat{x}) = \pi$. Both bounds also collapse to π : $\Lambda(1) = \lambda(1) = \pi$. The statement of the lemma is true.

Step 2.1: absolutely continuous consumer strategy. Now suppose that $-\gamma < D_o < 1$ and D is absolutely continuous on $(0, \hat{x})$. By Lemma 3, **q** is continuous at \hat{x} . Then, by Lemma 8, **q** is sustained by (\mathbf{p}, D, γ) over $[0, \hat{x}]$ if and only if:

$$\frac{\mathbf{q}(\hat{x})}{1-\mathbf{q}(\hat{x})} = W\left(\frac{\pi}{1-\pi} \exp\left[\frac{\pi}{1-\pi} - \frac{D(\hat{x})}{(1-\mathbf{q}(\hat{x}))S_m(\hat{x})}\right]\right)$$

As all locations below \hat{x} are outlet locations *D*-a.s., then $S_m(\hat{x}) = \int_{\hat{x}}^1 dD(y) + \gamma$. Rewriting the above equation, we obtain:

$$1 + \gamma = (D_o + \gamma)(1 - \mathbf{q}(\hat{x})) \left[\ln\left(\frac{\pi}{1 - \pi} \frac{1 - \mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})}\right) + \frac{1}{1 - \pi} \right]$$

Every absolutely continuous consumer strategy induces the same outlet quality composition for a given mass of outlet shoppers D_o .

Step 2.2: uniqueness of outlet quality. I now show that if \mathbf{q}_1 and \mathbf{q}_2 are both sustained by (\mathbf{p}, D, γ) , then $\mathbf{q}_1(\hat{x}) = \mathbf{q}_2(\hat{x})$.

Suppose, by way of contradiction, that $\mathbf{q}_1(\hat{x}) > \mathbf{q}_2(\hat{x})$ (the other case is symmetric). By Lemma 3, \mathbf{q}_1 and \mathbf{q}_2 are continuous at \hat{x} . Hence, there exists a left neighborhood of \hat{x} , such that $\mathbf{q}_1(\cdot) > \mathbf{q}_2(\cdot)$ for all stores within such neighborhood. Find the biggest such neighborhood, and define:

$$x_1 = \inf\{x < \hat{x} : \mathbf{q}_1(x) > \mathbf{q}_2(x)\}.$$

It must be that at x_1 , the downstream sales volume is higher in the first market outcome, $m_1 = (\mathbf{p}, D, \mathbf{q}_1, \gamma)$ than in the second one, $m_2 = (\mathbf{p}, D, \mathbf{q}_2, \gamma)$:

$$S_{m_2}(x_1) = D_o + \gamma + \int_{x_1}^{\hat{x}} \mathbf{q}_2(x) dD(x) < D_o + \gamma + \int_{x_1}^{\hat{x}} \mathbf{q}_1(x) dD(x) = S_{m_1}(x_1)$$

By Lemma 4, \mathbf{q}_1 and \mathbf{q}_2 being sustained by (\mathbf{p}, D, γ) requires:

$$(1 - \mathbf{q}_2(x_1)) = (1 - \mathbf{q}_2(\hat{x})) \frac{D_o + \gamma}{S_{m_2}(x_1)} > (1 - \mathbf{q}_2(\hat{x})) \frac{D_o + \gamma}{S_{m_1}(x_1)}$$

$$> (1 - \mathbf{q}_1(\hat{x})) \frac{D_o + \gamma}{S_{m_1}(x_1)} = (1 - \mathbf{q}(x_1))$$

Hence, $\mathbf{q}_2(x_1) < \mathbf{q}_1(x_1)$, and, by the same reasoning: $\mathbf{q}_2(x_1-) < \mathbf{q}_1(x_1-)$. Then, it must be that $x_1 = 0$ (or else x_1 is not correctly defined) and $\pi = \mathbf{q}_1(0) > \mathbf{q}_2(0) = \pi$, we obtain a contradiction.

Step 2.3: approximating continuous consumer strategies. Whenever D is continuous, we can approximate it with some sequence of absolutely continuous cdf, $\{D^n\}_{n=1}^{\infty}$. For each of these, we know how to construct an induced steady-state from Step 2.1.

I now go over this formally and show that if D is continuous on $[x_1, x_2] \subset [0, \hat{x}]$, then:

$$\frac{\mathbf{q}(x_2)}{1-\mathbf{q}(x_2)} = W\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\exp\left[\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)} - \frac{D(x_2) - D(x_1)}{(D_o + \gamma)(1-\mathbf{q}(\hat{x}))}\right]\right)$$

To simplify notation, denote $q^o = \mathbf{q}(\hat{x})$ in the sorting equilibrium $(\mathbf{p}, D, \mathbf{q}, \gamma)$.

Consider a sequence of shopping strategies such that each D^n admits a density almost everywhere on $[x_1, x_2]$ (for instance, take D^n to be piece-wise uniform) and that converges to $D(\cdot)$ (pointwisely) on $[x_1, x_2]$.

Construct respective sequence of quality compositions on $[x_1, x_2]$, $\mathbf{q}^n(\cdot)$, so that for every $x \in [x_1, x_2]$:

$$\left(S_m(x_1) - \int_{x_1}^x \mathbf{q}^n(x) dD^n(y)\right) (1 - \mathbf{q}^n(x)) = (1 - q^o)(D_o + \gamma).$$

For each D^n , $\mathbf{q}^n(x)$ is itself absolutely continuous and is given by:

$$\frac{\mathbf{q}^{n}(x)}{1-\mathbf{q}^{n}(x)} = W\left(\frac{\mathbf{q}(x_{1})}{1-\mathbf{q}(x_{1})}\exp\left[\frac{\mathbf{q}(x_{1})}{1-\mathbf{q}(x_{1})} - \frac{D^{n}(x) - D^{n}(x_{1})}{(D_{o}+\gamma)(1-q^{o})}\right]\right)$$

In addition, since D^n converges to D on $[x_1, x_2]$, then $\mathbf{q}^n(x)$ converges (pointwisely) to $\tilde{\mathbf{q}}$ with:

$$\frac{\tilde{\mathbf{q}}(x)}{1-\tilde{\mathbf{q}}(x)} = W\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\exp\left[\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)} - \frac{D(x)-D(x_1)}{(D_o+\gamma)(1-q^o)}\right]\right)$$

I now show that $(1 - \tilde{\mathbf{q}}(x))(S_m(x_1) - \int_{x_1}^x \tilde{\mathbf{q}}(y)dD(y))$ remains constant over $[x_1, x_2]$ and equals $(1 - q^o)(D_o + \gamma)$. Given the definition of \mathbf{q}^n , it suffices to show that:

$$\left(S_m(x_1) - \int_{x_1}^x \mathbf{q}^n(y) dD^n(y)\right) (1 - \mathbf{q}^n(x)) \xrightarrow[n \to \infty]{} (S_m(x_1) - \int_{x_1}^x \tilde{\mathbf{q}}(y) dD(y)) (1 - \tilde{\mathbf{q}}(x)),$$
$$\forall x \in [x_1, x_2].$$

Since $[x_1, x_2]$ is compact, D is uniformly continuous on $[x_1, x_2]$ by Heine–Cantor theorem. Hence, D^n converges to D uniformly. Since $\mathbf{q}(x_1)$ is bounded by π , and $D_o + \gamma > 0$ in any market outcome with positive sales, the argument inside W is bounded. Then, W is uniformly continuous on [0, K] for some K large enough and $\mathbf{q}^n(x)$ converges uniformly to $\tilde{\mathbf{q}}$.

It remains to verify that $\int_{x_1}^x \mathbf{q}^n(y) dD^n(y)$ converges to $\int_{x_1}^x \tilde{\mathbf{q}}(y) dD(y)$ for every x. We have:

$$\int_{x_1}^x \mathbf{q}^n(y) dD^n(y) = \int_{x_1}^x \tilde{\mathbf{q}}(y) dD^n(y) + \int_{x_1}^x \mathbf{q}^n(y) - \tilde{\mathbf{q}}(y) dD^n(y).$$

As as $\tilde{\mathbf{q}}$ is continuous on $[x_1, x_2]$, by Portmanteau theorem:

$$\int_{x_1}^x \tilde{\mathbf{q}}(y) dD^n(y) \xrightarrow[n \to \infty]{} \int_{x_1}^x \tilde{\mathbf{q}}(y) dD(y).$$

Hence, it now only remains to show that $\int_{x_1}^x \mathbf{q}^n(y) - \tilde{\mathbf{q}}(y) dD^n(y)$ converges to 0. As $\mathbf{q}^n(y)$ converges to $\tilde{\mathbf{q}}(y)$ uniformly, then for any $\varepsilon > 0$ for sufficiently large n:

$$\varepsilon \ge \varepsilon \int_{x_1}^x dD^n(y) \ge \int_{x_1}^x \mathbf{q}^n(y) - \tilde{\mathbf{q}}(y) dD^n(y) \ge -\varepsilon \int_{x_1}^x dD^n(y) \ge -\varepsilon$$

Taking $\varepsilon \to 0$, we get the desired convergence.

By the same proof as in the previous step, a sustained quality composition is uniquely defined at x_2 : $\mathbf{q}(x_2) = \tilde{\mathbf{q}}(x_2)$.

We may now conclude that if D has no atoms on $(0, \hat{x})$, then $\mathbf{q}(\hat{x}-) = \lambda(D_o, \gamma)$. By Lemma 3, \mathbf{q} is continuous at \hat{x} , so that

$$\mathbf{q}(\hat{x}-) = \mathbf{q}(\hat{x}) = \lambda(D_o, \gamma).$$

Step 3. Finally, let me show that the quality composition is within the suggested boundaries for every shopping strategy D.

Using Lemma 3 part (*iii*), if $\delta(\tilde{x}) > 0$ for some $\tilde{x} < \hat{x}$:

$$\frac{\mathbf{q}(\tilde{x}-)-\mathbf{q}(\tilde{x})}{(1-\mathbf{q}(\tilde{x}-))} = \frac{\delta(\tilde{x})\mathbf{q}(\tilde{x})}{S_m(\tilde{x})}$$
$$\frac{\mathbf{q}(\tilde{x}-)-\mathbf{q}(\tilde{x})}{(1-\mathbf{q}(\tilde{x}-))\mathbf{q}(\tilde{x})} = \frac{\delta(\tilde{x})}{S_m(\tilde{x})}$$
(6)

First, there is a unique visited non-outlet location $\tilde{x} \in (0, \hat{x})$. Then, by Lemma 4: $\mathbf{q}(\tilde{x}-) = \mathbf{q}(0) = \pi$. As all locations $(\tilde{x}, 1)$ are outlets (*D*-a.s.), then $S_m(\tilde{x}) = D^o + \gamma$. Plugging these into Equation (6), we obtain:

$$\frac{\pi - \mathbf{q}(\tilde{x})}{(1 - \pi)\mathbf{q}(\tilde{x})} = \frac{1 - D_o}{D_o + \gamma},$$

which simplifies to:

$$\mathbf{q}(\tilde{x}) = \pi \frac{D_o + \gamma}{D_o + \gamma + (1 - D_o)(1 - \pi)}$$

As $(\tilde{x}, 1]$ are *D*-a.s. outlet locations, we obtain:

$$\mathbf{q}(\hat{x}) = \mathbf{q}(\tilde{x}) = \Lambda(D_o, \gamma).$$

Now, consider any consumer strategy D. For every two non-outlet locations $x_1, x_2 < \hat{x}$, by Lemma 4 part (*ii*):

$$(1 - \mathbf{q}(x_1))S_m(x_1) = (1 - \mathbf{q}(x_2))S_m(x_2)$$

Locations $(0, \hat{x})$ are non-outlets, hence:

$$S_m(x_1) = S_m(x_2) + \int_{y \in (x_1, x_2]} \mathbf{q}(y) dD(y),$$

and we obtain:

$$(1 - \mathbf{q}(x_1)) \left(S_m(x_2) + \int_{y \in (x_1, x_2]} \mathbf{q}(y) dD(y) \right) = (1 - \mathbf{q}(x_2)) S_m(x_2),$$

which simplifies to:

$$\frac{\mathbf{q}(x_1) - \mathbf{q}(x_2)}{(1 - \mathbf{q}(x_1))} = \frac{\int_{y \in (x_1, x_2]} \mathbf{q}(y) dD(y)}{S_m(x_2)} \ge \frac{\mathbf{q}(x_2) \left(D(x_2) - D(x_1)\right)}{S_m(x_2)}$$

where we used $\mathbf{q}(\cdot)$ being non-increasing on $(0, \hat{x})$ by Lemma 4 part (i).

In particular, taking $x_1 = 0$ and $x_2 \to \hat{x}_-$, we obtain:

$$\frac{\pi - \mathbf{q}(\hat{x}-)}{1-\pi} \ge \frac{\mathbf{q}(\hat{x}-)(1-D_o)}{D_o + \gamma}.$$

As $\mathbf{q}(\cdot)$ is continuous at \hat{x} by Lemma 3 part (*ii*), we get from the above:

$$\mathbf{q}(\hat{x}) = \mathbf{q}(\hat{x}-) \le \pi \frac{D_o + \gamma}{D_o + \gamma + (1-\pi)(1-D_o)} = \Lambda(D_o, \gamma).$$

This confirms the upper bound on $\mathbf{q}(\hat{x})$.

Now, we move on to verifying the lower bound. By Step 2, whenever D is continuous on $[x_1, x_2)$:

$$\ln\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\frac{1-\mathbf{q}(x_2-)}{\mathbf{q}(x_2-)}\right) + \frac{1}{1-\mathbf{q}(x_1)} - \frac{1}{1-\mathbf{q}(x_2-)} = \frac{D(x_2-)-D(x_1)}{(D_o+\gamma)(1-q^o)}.$$

If there is a jump at x_2 , then using Equation (6) and replacing $S_m(x_2)$ with $(D_o + \gamma)(1 - q^o)/(1 - \mathbf{q}(x_2))$ (due to Lemma 4 (i)), we get that the overall change over an interval $[x_1, x_2]$ is:

$$\ln\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\frac{1-\mathbf{q}(x_2-)}{\mathbf{q}(x_2-)}\right) + \frac{1}{1-\mathbf{q}(x_1)} - \frac{1}{1-\mathbf{q}(x_2-)} + \frac{\mathbf{q}(x_2-)-\mathbf{q}(x_2)}{(1-\mathbf{q}(x_2-))\mathbf{q}(x_2)}\frac{1}{1-\mathbf{q}(x_2)} = \frac{D(x_2)-D(x_1)}{(D_o+\gamma)(1-q^o)}$$
(7)

Due to its concavity, $\ln(\cdot)$ satisfies:

$$\ln(y) < y - 1, \quad \forall y > 1.$$

When D is discontinuous at x_2 , so is \mathbf{q} : $\mathbf{q}(x_2-) \neq \mathbf{q}(x_2)$ so that we have:

$$\ln\left(\frac{\mathbf{q}(x_2-)}{1-\mathbf{q}(x_2-)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)}\right) < \frac{\mathbf{q}(x_2-)}{1-\mathbf{q}(x_2-)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)} - 1$$

Adding $\frac{1}{1-\mathbf{q}(x_2-)} - \frac{1}{1-\mathbf{q}(x_2)}$ to both sides of the inequality, we obtain a bound:

$$\ln\left(\frac{\mathbf{q}(x_{2}-)}{1-\mathbf{q}(x_{2}-)}\frac{1-\mathbf{q}(x_{2})}{\mathbf{q}(x_{2})}\right) + \frac{1}{1-\mathbf{q}(x_{2}-)} - \frac{1}{1-\mathbf{q}(x_{2})}$$
$$< \frac{\mathbf{q}(x_{2}-)}{1-\mathbf{q}(x_{2}-)}\frac{1-\mathbf{q}(x_{2})}{\mathbf{q}(x_{2})} - 1 + \frac{1}{1-\mathbf{q}(x_{2}-)} - \frac{1}{1-\mathbf{q}(x_{2})}$$
$$= \frac{\mathbf{q}(x_{2}-)-\mathbf{q}(x_{2})}{(1-\mathbf{q}(x_{2}-))\mathbf{q}(x_{2})}\frac{1}{1-\mathbf{q}(x_{2})}$$

We can now replace the right-hand side using Equation (7) to obtain:

$$\ln\left(\frac{\mathbf{q}(x_{2}-)}{1-\mathbf{q}(x_{2}-)}\frac{1-\mathbf{q}(x_{2})}{\mathbf{q}(x_{2})}\right) + \frac{1}{1-\mathbf{q}(x_{2}-)} - \frac{1}{1-\mathbf{q}(x_{2})}$$

$$< \frac{D(x_{2}) - D(x_{1})}{(D_{o}+\gamma)(1-q^{o})} - \ln\left(\frac{\mathbf{q}(x_{1})}{1-\mathbf{q}(x_{1})}\frac{1-\mathbf{q}(x_{2}-)}{\mathbf{q}(x_{2}-)}\right) - \frac{1}{1-\mathbf{q}(x_{1})} + \frac{1}{1-\mathbf{q}(x_{2}-)}$$

which simplifies to:

$$\ln\left(\frac{\mathbf{q}(x_1)}{1-\mathbf{q}(x_1)}\frac{1-\mathbf{q}(x_2)}{\mathbf{q}(x_2)}\right) + \frac{1}{1-\mathbf{q}(x_1)} - \frac{1}{1-\mathbf{q}(x_2)} < \frac{D(x_2) - D(x_1)}{(D_o + \gamma)(1-q^o)}.$$
(8)

If D is continuous on $(0, \hat{x})$, by Step 2, the sorting equilibrium m has outlet quality composition that achieves the lower bound, $\lambda(D_o, \gamma)$. Else, for finitely many discontinuities, we can find $\{x_1, \ldots, x_n\}$ of non-outlet locations, such that D is discontinuous at x_i . For each $[0, x_1], \ldots [x_i, x_{i+1}], [x_n, \hat{x}]$ Inequality (8) holds. Summing these up, we obtain:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o}{q^o}\right) + \frac{1}{1-\pi} - \frac{1}{1-q^o} < \frac{1-D_o}{(D_o+\gamma)(1-q^o)}$$

Part (iii) follows.

D Omitted Proofs for Section 4.1

Proof of Lemma 1. Follows from a more general Lemma 4. $\hfill \Box$

Proof of Theorem 1. The threshold structure of a sorting equilibrium in a binary-quality model follows from a more general Proposition 7.

Part (*i*). Suppose no outlets are visited. In a model with no direct disposal, by Lemma 5, if the measure of visited outlets is zero, then the equilibrium sales are zero. Consequently, both the seller and consumers get a zero equilibrium payoff.

Part (*ii*). Suppose all visited locations are outlets. By Proposition 1, if consumers visit outlets with probability one, then the sorting equilibrium is neutral. As consumers shop at prices of at most v^l , it follows that the consumer's payoff is at least $\pi(v^h - v^l)$.

Part (*iii*). Consider consumer payoff in a sorting equilibrium, where both types of locations are visited. By Lemma 4, all outlets have the same quality as the outlet threshold \hat{x} , *D*-a.s.. As a positive measure of outlets visited, the consumer's payoff is at least $\mathbf{q}(\hat{x})(v^h - v^l)$. By Proposition 7 part (*iii*), if non-outlets are visited with positive probability, consumers shop at prices (weakly) above v^l with probability 1. Together, these two imply consumer payoff

is exactly $\mathbf{q}(\hat{x})(v^h - v^l)$. The expression for the total surplus is derived in the main text. It follows from Lemma 2 (proven in the main text) and Proposition 1.

Proof of Proposition 1. By Lemma 8, for every $x \in (0, \hat{x})$

$$\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)} + \ln\left(\frac{\mathbf{q}(x)}{1-\mathbf{q}(x)}\right) = \ln\left(\frac{\pi}{1-\pi}\right) + \frac{\pi}{1-\pi} - \frac{\int_0^x \sigma(y)dy}{(1-\mathbf{q}(\hat{x}))\int_{\hat{x}}^1 \sigma(x)dx}$$

By Lemma 5, if the sales are positive, then $S_m(\hat{x}) > 0$. Then, by Lemma 3 (i), $\mathbf{q}(\cdot)$ is continuous at \hat{x} .

$$\frac{\mathbf{q}(\hat{x})}{1-\mathbf{q}(\hat{x})} + \ln\left(\frac{\mathbf{q}(\hat{x})}{1-\mathbf{q}(\hat{x})}\right) = \ln\left(\frac{\pi}{1-\pi}\right) + \frac{\pi}{1-\pi} - \frac{\int_{\hat{x}}^{1} \sigma(x)dx}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))} \\ = \ln\left(\frac{\pi}{1-\pi}\right) + \frac{\pi}{1-\pi} - \frac{1-S_m(\hat{x})}{S_m(\hat{x})(1-\mathbf{q}(\hat{x}))}$$

Replacing $S_m(\hat{x})$ with $S_m(0)\frac{1-\pi}{1-\mathbf{q}(\hat{x})}$ by Lemma 4 part (*i*), we obtain:

$$\frac{\mathbf{q}(\hat{x})}{1-\mathbf{q}(\hat{x})} + \ln\left(\frac{\mathbf{q}(\hat{x})}{1-\mathbf{q}(\hat{x})}\right) = \ln\left(\frac{\pi}{1-\pi}\right) + \frac{\pi}{1-\pi} - \frac{1}{S_m(0)(1-\pi)} + \frac{1}{1-\mathbf{q}(\hat{x})}$$

Rearranging, we get the desired expression connecting total sales volume and sorting precision:

$$S_m(0) \left[\ln \left(\frac{\pi}{1-\pi} \frac{1-\mathbf{q}(\hat{x})}{\mathbf{q}(\hat{x})} \right) (1-\pi) + 1 \right] = 0$$

Sorting Precision Implementation. To finalize the proof of the theorem, Lemma 10 constructs a sorting equilibrium with any given outlet quality $q \in (0, \pi]$.

Lemma 10. For any $q \in (0, \pi]$, there exists a sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q}) \in \mathcal{E}$ that has quality q at the outlet threshold \hat{x} : $\mathbf{q}(\hat{x}) = q$.

Proof. Take any $q \in (0, \pi]$. I now construct a sorting equilibrium $(\mathbf{p}, \sigma, \mathbf{q})$, such that $\mathbf{q}(\hat{x}) = q$. Take $\sigma(x) = 1, \forall x \in X$. Compute the outlet threshold \hat{x} from:

$$(1-q)\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + \frac{1}{1-\pi}\right) = \frac{1}{1-\hat{x}}$$

Note that for any $q \in (0, \pi]$, the outlet threshold is interior $\hat{x} \in [0, 1)$. Set all locations $[\hat{x}, 1]$ to be outlets with price v^l : $\mathbf{p}(\cdot) = v^l$. Define the quality composition to be $\mathbf{q}(x) = q, \forall x \ge \hat{x}$. Then by Lemma 1, $\mathbf{q}(\cdot)$ is sustained by (\mathbf{p}, σ) on $[\hat{x}, 1]$.

For earlier locations $x \in [0, \hat{x})$, let $\mathbf{q}(\cdot)$ be defined by:

$$\frac{\mathbf{q}(x)}{1 - \mathbf{q}(x)} = W\left(\frac{\pi}{1 - \pi} \exp\left[\frac{\pi}{1 - \pi} - \frac{x}{(1 - \hat{x})(1 - q)}\right]\right)$$

Define prices from $\mathbf{p}(x) = v^h - \frac{q}{\mathbf{q}(x)}(v^h - v^l), \forall x \in [0, \hat{x})$. By construction, $\mathbf{q}(\cdot)$ is continuous at \hat{x} and is sustained by (\mathbf{p}, σ) on $[1, \hat{x}]$ by Lemma 8 since $\mathbf{p}(\cdot) > v^l$ on $(0, \hat{x})$.

Finally, the prices make consumers indifferent between all locations, and σ is optimal given (\mathbf{p}, \mathbf{q}) .

E Omitted Proofs for Section 4.3

In this appendix, I analyze the properties of the seller's payoff as a function of the outlet quality composition:

$$\tilde{V}^{S}(q) = \frac{\pi v^{h} + (1 - \pi)v^{l}}{(1 - \pi)\ln\left(\frac{\pi}{1 - \pi}\frac{1 - q}{q}\right) + 1} - q(v^{h} - v^{l}).$$

Lemma 11. $\tilde{V}^{S}(\cdot)$ has the following properties:

- (i) $\tilde{V}^S(\pi) = v^l$
- (*ii*) $\partial_q \tilde{V}^S(\pi) > 0$
- (iii) $\lim_{q \to 0} \tilde{V}^S(q) = 0$, $\lim_{q \to 0} \partial_q \tilde{V}^S = \infty$
- (iv) \tilde{V}^S is concave-convex: that is, there exists $\bar{q}(\pi) \in (0, \pi/2)$, such that \tilde{V}^S is convex on $(\bar{q}(\pi), \pi]$ and is concave on $[0, \bar{q}(\pi))$

Proof. Before proving part (i) - (iv), consider the derivative of \tilde{V}^S :

$$\partial_q \tilde{V}^S(q) = \frac{\pi v^h + (1-\pi)v^l}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + 1\right)^2} \frac{1-\pi}{(1-q)q} - (v^h - v^l) \tag{9}$$

(i) Is straightforward from plugging in $q = \pi$ into $\tilde{V}^{S}(\cdot)$.

(*ii*) Plugging in $q = \pi$ into Equation (9), we get:

$$\partial_q \tilde{V}^S(\pi) = \frac{\pi v^h + (1 - \pi)v^l}{\pi} - (v^h - v^l) = \frac{v^l}{\pi} > 0$$

(iii)

$$\lim_{q \to 0} \tilde{V}^S(q) = \lim_{q \to 0} \frac{\pi v^h + (1 - \pi)v^l}{(1 - \pi)\ln\left(\frac{\pi}{1 - \pi}\frac{1 - q}{q}\right) + 1} = 0$$

$$\lim_{q \to 0} \partial_q \tilde{V}^S = \lim_{q \to 0} \frac{\pi v^h + (1 - \pi) v^l}{\left((1 - \pi) \ln\left(\frac{\pi}{1 - \pi} \frac{1 - q}{q}\right) + 1 \right)^2} \frac{1 - \pi}{(1 - q)q} - (v^h - v^l)$$

I now compute the limit of $\lim_{q\to 0} \frac{1}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2} \frac{1}{(1-q)q}$ by applying L'Hôpital's rule twice:

$$\lim_{q \to 0} \frac{1/[(1-q)q]}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1\right)^2} = \lim_{q \to 0} \frac{1}{2(1-\pi)} \frac{(1-2q)/[(1-q)q]}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right)+1}$$
$$= \lim_{q \to 0} \frac{1}{2(1-\pi)^2} \frac{1-2q+2q^2}{(1-q)q} = \infty$$

(iv)

$$\partial_{qq}^{2}\tilde{V}^{S}(q) = \frac{\pi v^{h} + (1-\pi)v^{l}}{(1-q)^{2}q^{2}} \frac{2(1-\pi) - (1-2q)\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + 1\right)}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + 1\right)^{3}}$$

The sign of $\partial^2_{qq} \tilde{V}^S(q)$ is then determined by the sign of:

$$2(1-\pi) - (1-2q)\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q}{q}\right) + 1\right)$$
(10)

Whenever $q \in [\min\{1/2, \pi\}, \pi]$, the above is positive. For q < 1/2, the Expression (10) is increasing with q, and is negative at $q \to 0$. Then, Expression (10) crosses zero exactly once. Denote the zero of Expression (10) as $\bar{q}(\pi)$. To get a more precise bound

on \bar{q} , use $\ln(y) \le y - 1$ for all y > 1:

$$2 = (1 - 2\bar{q}(\pi)) \left(\ln\left(\frac{\pi}{1 - \pi} \frac{1 - \bar{q}(\pi)}{\bar{q}(\pi)}\right) + \frac{1}{1 - \pi} \right) \le (1 - 2\bar{q}(\pi)) \frac{\pi}{1 - \pi} \frac{1}{\bar{q}(\pi)},$$

which simplifies to $\bar{q}(\pi) \leq 0.5\pi$ for $\pi \in (0, 1)$.

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For any (π, v^h) , where the seller prefers an interior solution, it is given by:

$$\partial_q \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) = 0$$

$$\partial^2_{q \ q} \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) < 0$$

By Lemma 11, $q^o(\pi, v^h)$ is unique at (π, v^h) .

Next, Proposition 8 shows when the seller's optimal solution is a neutral sorting equilibrium. This occurs when high-quality goods are either not valued enough by the consumers or when they are too rare at the production plant.

Proposition 8. There exist $\bar{\pi}(v^h, v^l)$ and $\bar{v}^h(\pi, v^l)$, such that \tilde{V}^S attains its maximum at $q = \pi$ if and only if $\pi \leq \bar{\pi}(v^h, v^l)$, $v^h \leq \bar{v}^h(\pi, v^l)$.

Proof. Step 1. First, I establish that it is optimal for the seller to choose a neutral sorting equilibrium whenever $\pi \to 0$ or $v^h \to v^l$; but to choose an active sorting equilibrium for $\pi \to 1$ or $v^h \to \infty$.

For any (π, v^h) , we have boundary on the seller's payoff $\tilde{V}^S_{(\pi, v^h)}(\cdot)$:

$$\tilde{V}^{S}_{(\pi,v^{h})}(q) \le \pi v_{h} + (1-\pi)v^{l} - q(v^{h} - v^{l})$$

with a strict inequality for any $q < \pi$. As $\pi \to 0$ or $v^h \to v^l$, this boundary converges to v^l for every feasible q. The seller can actually achieve a payoff of v^l by doing no sorting.

 $\pi \to 1$: Fix a sorting precision, so that the average quality at the outlets is $\frac{1-\pi}{\pi}$. For sufficiently large π , $\frac{1-\pi}{\pi}\pi$ and hence is feasible. As $\pi \to 1$, we obtain the following boundary on the seller's payoff from some interior solution:

$$\max_{q \in (0,\pi)} \tilde{V}^{S}_{(\pi,v^{h})}(q) \ge \tilde{V}^{S}_{(\pi,v^{h})}\left(\frac{1-\pi}{\pi}\right)$$
$$= \left(\pi v^{h} + (1-\pi)v^{l}\right) / \left(2\ln\left(\frac{\pi}{1-\pi}\right)(1-\pi) + 1\right) - \frac{1-\pi}{\pi}(v^{h} - v^{l})$$

$$\underset{\pi \to 1}{\to} v^h > v^l$$

 $v^h \to \infty$: Similarly, fix any outlet quality composition at $\frac{v^l}{v^h - v^l}$, which is feasible for large enough v^h . Then, the seller's payoff from the interior solution is at least:

$$\max_{q \in (0,\pi)} \tilde{V}^{S}_{(\pi,v^{h})}(q_{o}) \geq \tilde{V}^{S}_{(\pi,v^{h})}\left(\frac{v^{l}}{v^{h} - v^{l}}\right) = \frac{\pi v^{h} + (1 - \pi)v^{l}}{(1 - \pi)\ln\left(\frac{\pi}{1 - \pi}\frac{v^{h} - 2v^{l}}{v^{h} - v^{l}}\right) + 1} - v^{l}$$
$$\xrightarrow[v^{h} \to \infty]{} \infty > v^{l}$$

This completes Step 1. From Step 1, there exist some parameters where the optimum switches between some interior solution and the corner solution at $q = \pi$.

Step 2. Now, I show that switching between the two solution types can only happen once.

I first show this for the probability of high-quality goods, π . Suppose that at (π, v^h) , the seller's maximal payoff is strictly above v^h .

By the Envelope Theorem,

$$\frac{d\tilde{V}^{S}_{(\pi,v^{h})}(q^{o}(\pi,v^{h}))}{d\pi} = \partial_{\pi}\tilde{V}^{S}_{(\pi,v^{h})}(q^{o}(\pi,v^{h})),$$

where

$$\partial_{\pi} \tilde{V}^{S}_{(\pi,v^{h})}(q) = \frac{v^{h} - v^{l}}{(1 - \pi) \ln\left(\frac{\pi}{1 - \pi} \frac{1 - q}{q}\right) + 1} + \left(\pi v^{h} + (1 - \pi)v^{l}\right) \frac{\ln\left(\frac{\pi}{1 - \pi} \frac{1 - q}{q}\right) - \frac{1}{\pi}}{\left((1 - \pi) \ln\left(\frac{\pi}{1 - \pi} \frac{1 - q}{q}\right) - \frac{v^{l}}{\pi}\right)}$$
$$= \frac{v^{h} \ln\left(\frac{\pi}{1 - \pi} \frac{1 - q}{q}\right) - \frac{v^{l}}{\pi}}{\left((1 - \pi) \ln\left(\frac{\pi}{1 - \pi} \frac{1 - q}{q}\right) + 1\right)^{2}}.$$

Then, the sign of $\frac{d\tilde{V}^{S}_{(\pi,v^{h})}(q^{o}(\pi,v^{h}))}{d\pi}$ coincides with the sign of

$$v^{h} \ln\left(\frac{\pi}{1-\pi} \frac{1-q^{o}(\pi, v^{h}))}{q^{o}(\pi, v^{h})}\right) - \frac{v^{l}}{\pi}.$$
(11)

If the seller ever switches to the interior solution (which is true by Step 1), then $\frac{d\tilde{V}^{S}_{(\pi,v^{h})}(q^{o}(\pi,v^{h}))}{d\pi} > 0$ is positive at least some (π_{1}, v^{h}) . Let me show that $\frac{d\tilde{V}^{S}_{(\pi,v^{h})}(q^{o}(\pi,v^{h}))}{d\pi} \ge 0$ at all (π, v^{h}) where $\pi > \pi_{1}$. It is enough to establish that the Expression (11) is increasing in π at any (π, v^{h}) where Expression (11) equals 0. In turn, it suffices to show that $q^{o}(\pi, v^{h})$ is decreasing in π

at any such (π, v^h) .

By definition of $q^o(\pi, v^h)$, it is decreasing in π at (π, v^h) whenever $\partial_{q\pi}^2 \tilde{V}^S_{(\pi, v^h)}(q^o(\pi, v^h)) < 0$.

$$\begin{aligned} \partial_{q\pi}^{2} \tilde{V}_{(\pi,v^{h})}^{S}(q^{o}(\pi,v^{h})) &\propto \frac{(1-\pi)(v^{h}-v^{l})}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)+1\right)^{2}} \\ &- \frac{\pi v^{h} + (1-\pi)v^{l}}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)+1\right)^{2}} \\ &+ 2(1-\pi)\frac{(\pi v^{h} + (1-\pi)v^{l})\left(\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)-\frac{1}{\pi}\right)}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)+1\right)^{3}} \\ &= (1-\pi)\frac{v^{h}\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)-\frac{v^{l}}{\pi}}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)+1\right)^{3}} \\ &- (\pi v^{h} + (1-\pi)v^{l})\frac{\frac{1}{\pi}}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right)+1\right)^{3}} \end{aligned}$$

Hence, if Expression (11) is 0 at (π, v^h) , then $\partial_{q\pi}^2 \tilde{V}^S_{(\pi, v^h)}(q^o(\pi, v^h)) < 0$. Consequently, Expression (11) is increasing in π at (π, v^h) .

Now, I do the same exercise for v^h . Suppose the seller strictly prefers the interior solution at some (π, v^h) , then $q^o(\pi, v^h)$ is well-defined and:

$$\frac{d\tilde{V}_{(\pi,v^h)}^S(q^o(\pi,v^h))}{dv^h} = \partial_{v^h}\tilde{V}_{(\pi,v^h)}^S(q^o(\pi,v^h)) = \frac{\pi}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o(\pi,v^h)}{q^o(\pi,v^h)}\right) + 1} - q^o(\pi,v^h)$$

As $\ln(y) \le y - 1$, we get a lower bound on the above:

$$\frac{d\tilde{V}^{S}_{(\pi,v^{h})}(q^{o}(\pi,v^{h}))}{dv^{h}} \ge q^{o}(\pi,v^{h}) - q^{o}(\pi,v^{h}) = 0$$

Hence, the seller's payoff from the interior solution is increasing in v^h . Then, if the seller prefers the interior solution at (π, v_1^h) , she strictly prefers an interior solution for all (π, v^h) with $v^h > v_1^h$.

By Proposition 8, the seller prefers an interior solution as long as the value of high-quality

goods, or their probability at the production plant, is sufficiently high.

Proof of Proposition 2. I now derive the comparative statics of q^o with respect to π and v^h . I restrict attention to parameters (π, v^h) where the seller strictly prefer the interior solution, and $q^o(\cdot, \cdot)$ is well-defined.

Step 1: comparative statics of q^o with respect to π . The sign of $\partial_{\pi}q^o(\pi, v^h)$ is determined by the sign of $\partial_{q,\pi}^2 \tilde{V}^S_{(\pi,v^h)}(q^o(\pi, v^h))$. By the algebra in proof of Proposition 8:

$$\begin{aligned} \partial_{q\pi}^{2} \tilde{V}_{(\pi,v^{h})}^{S}(q^{o}(\pi,v^{h})) &\propto \pi v^{h} - 2 \frac{\pi v^{h} + (1-\pi)v^{l}}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}(\pi,v^{h})}{q^{o}(\pi,v^{h})}\right) + 1} \\ &= \pi v^{h} - 2\tilde{V}_{(\pi,v^{h})}^{S}(q^{o}(\pi,v^{h})) + 2q^{o}(\pi,v^{h})(v^{h}-v^{l}) \end{aligned}$$

By Lemma 11, $q^o(\pi, v^h) < 0.5\pi$, and we can bound the above:

$$\begin{aligned} \pi v^h &- 2 \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) + 2 q^o(\pi,v^h)(v^h - v^l) \\ &< 2\pi v^h - 2 q^o(\pi,v^h) v^l - 2 \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) \end{aligned}$$

Using Step 1 in Proposition 8:

$$\lim_{\pi \to 1} \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) = v^h$$

Then, for high enough π , $\partial_{q\pi}^2 \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) < 0.$

Step 2: comparative statics of q^o with respect to v^h . At the interior candidate solution, the sign of $\partial_{v^h} q^o(\pi, v^h)$ is given by the sign of $\partial_{qv^h}^2 \tilde{V}^S_{(\pi, v^h)}(q^o(\pi, v^h))$:

$$\partial_{qv^h}^2 \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) = \frac{\pi}{\left((1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^o(\pi,v^h)}{q^o(\pi,v^h)}\right) + 1\right)^2} \frac{1-\pi}{(1-q^o(\pi,v^h))q^o(\pi,v^h)} - 1$$

Using the definition of q^{o} , we can replace the first summand in the above to get:

$$\partial_{qv^h}^2 \tilde{V}^S_{(\pi,v^h)}(q^o(\pi,v^h)) = \frac{\pi(v^h - v^l)}{\pi v^h + (1-\pi)v^l} - 1 = -\frac{v^l}{\pi v^h + (1-\pi)v^l} < 0$$

F Omitted proofs for Section 4.4

Proof of Proposition 3. By Proposition 7, a sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ is \hat{x} -threshold market outcome. For brevity, let $q^o \equiv \mathbf{q}(\hat{x})$.

By Lemma 7 the total surplus is given by:

$$TS(\mathbf{p},\sigma,\mathbf{q},\gamma) = \left(\int_{y\in[\hat{x},1)} \sigma(y)dy + \gamma\right) (1-q^o) \left(\frac{\pi}{1-\pi}v^h + v^l\right) - \gamma q^o(v^h - v^l) - \gamma(\kappa + v^l).$$

By Lemma 9, when consumer strategy is continuous in $(0, \hat{x})$, we have:

$$TS(\mathbf{p},\sigma,\mathbf{q},\gamma) = \left(\pi v^{h} + (1-\pi)v^{l}\right) \frac{1+\gamma}{(1-\pi)\ln\left(\frac{\pi}{1-\pi}\frac{1-q^{o}}{q^{o}}\right) + 1} - \gamma q^{o}(v^{h}-v^{l}) - \gamma(\kappa+v^{l}).$$

The seller's payoff is the difference between the total surplus and the consumer payoff:

$$V^{S}(\mathbf{p},\sigma,\mathbf{q},\gamma) = TS(\mathbf{p},\sigma,\mathbf{q},\gamma) - V^{B}(\mathbf{p},\sigma,\mathbf{q},\gamma).$$

Similar to Theorem 1, we now consider three cases of the sorting equilibria, depending on the consumer share of outlets.

Case 1: only non-outlets are visited. In this case, $\int_{y \in [\hat{x},1)} dD(y) = 0$, and the seller may extract the whole total surplus from a market outcome by charging a price of v^h at all store locations.

By Lemma 9, the quality composition at 1 is given by $\Phi(\gamma)$, where:

$$\left[\ln\left(\frac{\pi}{1-\pi}\frac{1-\Phi(\gamma)}{\Phi(\gamma)}\right) + \frac{1}{1-\pi}\right] = \frac{1+\gamma}{\gamma(1-\Phi(\gamma))}$$

Note that $\Phi(\gamma)$ is the lowest outlet quality composition that the seller could achieve for a fixed disposal rate γ . We can write the seller's profit for a sorting equilibrium with no outlets as:

$$\hat{V}^{S}_{\kappa}(\gamma) = \left(\frac{\pi}{1-\pi}v^{h}\gamma(1-\Phi(\gamma)) - \Phi(\gamma)v_{h}\gamma - \gamma\kappa\right).$$

Denote the seller's maximal profit from such sorting equilibria as V^{**} :

$$V^{**} = \sup_{\gamma > 0} \hat{V}^S(\gamma, \kappa).$$

Case 2: only outlets are visited. Whenever consumers shop only at outlet locations, the seller receives a constant price $\bar{p} \leq v^l$, and the seller's profit is at most v^l , which is equal to $\tilde{V}^S(\pi)$.

Case 3: both types of locations are visited. If consumers shop at both outlets and non-outlets,

then by Proposition 7, $V^B(\mathbf{p}, D, \mathbf{q}, \gamma) = q^o(v^h - v^l)$ and we get:

$$V^{S}(\mathbf{p}, D, \mathbf{q}, \gamma) = TS(\mathbf{p}, D, \mathbf{q}, \gamma) - V^{B}(\mathbf{p}, D, \mathbf{q}, \gamma)$$
$$= \left(\pi v^{h} + (1 - \pi)v^{l}\right) \frac{1 + \gamma}{(1 - \pi)\ln\left(\frac{\pi}{1 - \pi}\frac{1 - q^{o}}{q^{o}}\right) + 1}$$
$$- \gamma q^{o}(v^{h} - v^{l}) - \gamma(\kappa + v^{l}) - q^{o}(v^{h} - v^{l})$$
$$= (1 + \gamma)\tilde{V}^{S}(q^{o}) - \gamma(\kappa + v^{l})$$

Hence, the seller's maximal profit among all sorting equilibria where consumers shop at both types of locations is:

$$V^* = \sup_{\substack{\gamma \ge 0}} \sup_{\substack{q \in [\Phi(\gamma), \pi] \\ q > 0}} (1+\gamma) \tilde{V}^S(q) - \gamma(\kappa + v^l)$$

Clearly, $V^* \ge v^l = \tilde{V}^S(\pi)$, which the seller can achieve with no direct disposal by setting $\gamma = 0$. Then, the seller's optimal choice reduces to selecting between V^* and V^{**} .

Let q^* be the optimizer of \tilde{V}^S over $(0, \pi]$. I now establish the following:

$$\max\{V^*, V^{**}\} = \begin{cases} \max_{\gamma \in (0,\infty)} \hat{V}_{\kappa}^{S}(\gamma), \text{ if } \tilde{V}^{S}(q^*) > \kappa + v^l \\ \tilde{V}^{S}(q^*), \text{ if } \pi/(1-\pi)v^h \le \kappa \\ \max\left\{\tilde{V}^{S}(q^*), \max_{\gamma \in (0,\infty)} \hat{V}_{\kappa}^{S}(\gamma)\right\}, \text{ if } \kappa \in \left[\tilde{V}^{S}(q^*) - v^l, \frac{\pi}{1-\pi}v^h\right) \end{cases}$$

Case 1: $\tilde{V}^{S}(q^{*}) > \kappa + v^{l}$. Then, the seller is better off not having outlet locations: $V^{**} > V^{*}$. Note first that $\Phi(\gamma)$ is increasing γ and:

$$\lim_{\gamma \to 0} \Phi(\gamma) = 0, \ \lim_{\gamma \to \infty} \Phi(\gamma) = \pi.$$

If $\tilde{V}^{S}(q^{*}) > \kappa + v^{l}$, then, with no disposal, the seller's profit is maximized at an active sorting equilibrium: $q^{*} < \pi$. There exists a unique $\tilde{\gamma} \in (0, \infty)$, such that the seller achieves exactly the sorting precision that maximizes $\tilde{V}^{S}(\cdot)$:

$$\Phi(\tilde{\gamma}) = q^*.$$

For all lower disposal rates $\gamma \leq \tilde{\gamma}$ and all outlet qualities $q \in (0, \pi]$:

$$(1+\gamma)\tilde{V}^S(q) - \gamma(\kappa+v^l) \le (1+\gamma)\tilde{V}^S(q^*) - \gamma(\kappa+v^l)$$

$$\leq (1+\tilde{\gamma})\tilde{V}^{S}(q^{*}) - \tilde{\gamma}(\kappa + v^{l})$$

= $(1+\tilde{\gamma})\tilde{V}^{S}(\Phi(\tilde{\gamma})) - \tilde{\gamma}(\kappa + v^{l})$
< $\left(\frac{\pi}{1-\pi}v^{h}\tilde{\gamma}(1-\Phi(\tilde{\gamma})) - \Phi(\tilde{\gamma})v_{h}\tilde{\gamma} - \tilde{\gamma}\kappa\right) \leq V^{**}.$

Similarly, for all higher disposal rates $\gamma > \tilde{\gamma}$:

$$\sup_{\substack{q \in [\Phi(\gamma),\pi] \\ q > 0}} (1+\gamma)\tilde{V}^{S}(q) - \gamma(\kappa+v^{l}) = (1+\gamma)\max\{\tilde{V}^{S}(\pi), \tilde{V}^{S}(\Phi(\gamma))\} - \gamma(\kappa+v^{l}) < V^{**}.$$

where I use the fact that \tilde{V}^S is convex-concave by Lemma 11, and hence whenever the lower bound is binding $(q^* < \Phi(\gamma))$, \tilde{V}^S reaches its optimum at one of the corners. That is, $\tilde{V}^S(q^*) > \kappa + v^l$ is sufficient for the seller not to use outlet locations. **Case 2.** Alternatively, suppose $\tilde{V}^S(q^*) \le \kappa + v^l$, then:

$$\sup_{\substack{q \in [\Phi(\gamma),\pi]\\q>0}} (1+\gamma)\tilde{V}^S(q) - \gamma(\kappa+v^l) \le (1+\gamma)\tilde{V}^S(q^*) - \gamma(\kappa+v^l) \le \tilde{V}^S(q^*)$$

That is, in this case, $V^* = \tilde{V}^S(q^*)$. The seller does not use direct disposal simultaneously with outlet locations. Additionally, for $\pi/(1-\pi)v^h \leq \kappa$, the seller prefers outlets to be visited, since

$$V^* \ge \tilde{V}^S(\pi) = v^l > 0 = V^{**}.$$

Optimal Disposal Rate. Let me now verify that an optimal \hat{V}^S attains its optimum on $(0, \infty)$ for any $\kappa > 0$ whenever $\pi/(1-\pi)v^h > \kappa$. To that end, we examine the limit $\partial_{\gamma} \hat{V}^S_{\kappa}(\gamma)$ at the two corners.

$$\partial_{\gamma} \hat{V}_{\kappa}^{S}(\gamma) = \frac{\pi}{1-\pi} v^{h} (1-\Phi(\gamma)) - \Phi(\gamma) v_{h} - \kappa - \frac{\gamma v^{h}}{1-\pi} \partial_{\gamma} \Phi(\gamma)$$

An unbounded disposal rate is suboptimal:

$$\partial_{\gamma} \hat{V}^{S}_{\kappa}(\gamma) \xrightarrow[\gamma \to \infty]{} -\kappa - \lim_{\gamma \to \infty} \frac{\gamma v^{h}}{1 - \pi} \partial_{\gamma} \Phi(\gamma) \leq -\kappa$$

as $\partial_{\gamma} \Phi(\gamma) > 0$.

Consider now the other bound:

$$\partial_{\gamma} \hat{V}^{S}_{\kappa}(\gamma) \xrightarrow[\gamma \to 0]{} \frac{\pi}{1-\pi} v^{h} - \kappa - \lim_{\gamma \to 0} \frac{\gamma v^{h}}{1-\pi} \partial_{\gamma} \Phi(\gamma)$$

Then, to establish the optimal choice of γ is strictly above 0 for any $\pi/(1-\pi)v^h > \kappa$, it is enough to show $\lim_{\gamma \to 0} \gamma \partial_\gamma \Phi(\gamma) = 0$.

$$\gamma \partial_{\gamma} \Phi(\gamma) = \frac{1}{(1+\gamma)/(1-\Phi(\gamma)) + \gamma/\Phi(\gamma)}$$

Hence, the limit $\lim_{\gamma\to 0} \gamma \partial_{\gamma} \Phi(\gamma) = 0$ is determined by the limit of $\gamma/\Phi(\gamma)$. From the definition of Φ , it must be that γ converges to 0 at the same rate as $\ln(\Phi(\gamma))$, hence $\lim_{\gamma\to 0} \frac{\gamma}{\Phi(\gamma)} = \infty$ implying $\lim_{\gamma\to 0} \gamma \partial_{\gamma} \Phi(\gamma) = 0$ as required.

Regime Switches. Finally, to establish there is a unique threshold where the optimal regime switches, note that $\tilde{V}^{S}(q^{*})$ is independent of κ whereas $\max_{\gamma \in (0,\infty)} \hat{V}_{\kappa}^{S}(\gamma)$ is strictly decreasing in κ . Hence, for every parameters (v^{h}, v^{l}, π) , there exists $\bar{\kappa}$ as in the formulation of the proposition.

To prove $\bar{\kappa}$ is increasing in v^l , note that $\max_{\gamma \in (0,\infty)} \hat{V}^S_{\kappa}(\gamma)$ is constant in v^l , but $\tilde{V}^S(q^*)$ is strictly increasing in v^l . Hence, if v^l increases, the switch occurs at a lower production/disposal cost.

G Vintage-Based Pricing

I now provide details for the vintage-based pricing model.

The vintage-based sorting equilibrium requires both vintage distribution and quality composition to be in a steady state.

The total outflow of products with vintages in $(x_1, x_2]$ is: the purchases of these vintages, and the mass of vintage x_2 . The total inflow of goods equals the mass of products of vintage x_1 . Then, the total stock at vintages in $(x_1, x_2]$ stays the same when:

$$\int_{y \in [x_1, x_2]} \sigma(y) \left[\mathbf{q}(y) \mathbb{1}\{ \mathbf{p}(y) \le v^h \} + (1 - \mathbf{q}(y)) \mathbb{1}\{ \mathbf{p}(y) \le v^l \} \right] dy + \mu(x_2) = \mu(x_1)$$
(12)

Similarly, the mass of high-quality goods of vintages $(x_1, x_2]$ is preserved when:

$$\int_{y \in [x_1, x_2]} \sigma(y) \mathbf{q}(y) \mathbb{1}\{\mathbf{p}(y) \le v^h\} dy + \mathbf{q}(x_2) \mu(x_2) = \mathbf{q}(x_1) \mu(x_1)$$
(13)

We now say that (μ, \mathbf{q}) is sustained by (\mathbf{p}, σ) on $Y \subseteq X$ if both Equation (12) and Equation (13) hold for each $(x_1, x_2] \subseteq Y$. Say that (μ, \mathbf{q}) is sustained by (\mathbf{p}, σ) if it is sustained on [0, 1]. **Payoffs**. The payoffs in this alternative formulation of the model remain the same. They are described by the same functions as in the benchmark model. Consumer payoff in a vintagebased market outcome $(\mathbf{p}, \sigma, \mu, \mathbf{q})$ only depends on consumer strategy, prices, and quality composition and is given by $V^B(\mathbf{p}, \sigma, \mathbf{q})$. The seller's payoff only depends on the distribution of vintages through the rate of disposal. It is given by $V^S(\mathbf{p}, \sigma, \mathbf{q}, \mu(1))$.

Vintage-Based Sorting Equilibrium. We adjust the sorting equilibrium definition to account for the endogeneity of stock distribution across vintages. Say that a vintage-based market outcome $w = (\mathbf{p}, \sigma, \mu, \mathbf{q})$ is a *a vintage-based sorting equilibrium* if both stock distribution and quality composition (μ, \mathbf{q}) are sustained by prices and consumer strategy (\mathbf{p}, σ) , and σ maximizes consumer payoff $V^B(\mathbf{p}, \sigma, \mathbf{q})$ for the given (\mathbf{p}, \mathbf{q}) .

Vintage-Based Sorting Equilibrium. We adjust the sorting equilibrium definition to account for the endogeneity of stock distribution across vintages. Say that a market outcome $m = (\mathbf{p}, \sigma, \mu, \mathbf{q})$ is a *a vintage-based sorting equilibrium* if (*i*) both stock distribution and quality composition (μ, \mathbf{q}) are sustained by prices and consumer strategy (\mathbf{p}, σ) , and (*ii*) σ maximizes consumer payoff $V^B(\mathbf{p}, \sigma, \mathbf{q})$ for the given (\mathbf{p}, \mathbf{q}) .

Proof of Theorem 2. Consider a vintage-based market-outcome $w = (\mathbf{p}, \sigma, \mu, \mathbf{q})$, and a market outcome $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma)$ with disposal rate $\gamma = \mu(1)$. We verify that w is a vintage-based sorting equilibrium if and only if m is a sorting equilibrium.

Part (ii), consumer optimality, is the same across the two notions of equilibrium by definition.

We need only show that (μ, \mathbf{q}) is sustained by (\mathbf{p}, σ) in w if and only if \mathbf{q} is sustained by $(\mathbf{p}, \sigma, \gamma)$ in m.

By definition, (μ, \mathbf{q}) is sustained by (\mathbf{p}, σ) in w, whenever for every $x \in [0, 1]$:

$$\mu(x) = \mu(1) + \int_{y \in [x,1)} \sigma(y) \left[\mathbf{q}(y) \mathbb{1}\{\mathbf{p}(y) \le v^h\} + (1 - \mathbf{q}(y)) \mathbb{1}\{\mathbf{p}(y) \le v^h\} \right] dy$$

and

$$\mathbf{q}(x)\mu(x) = \mu(1)\mathbf{q}(1) + \int_{y \in [x,1)} \sigma(y)\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \le v^h\}dy$$

And **q** is sustained in $(\mathbf{p}, \sigma, \gamma)$ in *m* whenever for every $x \in [0, 1)$:

$$\mathbf{q}(x)S_m(x) = \gamma \mathbf{q}(1-) + \int_{y \in [x,1)} \sigma(y)\mathbf{q}(y)\mathbb{1}\{\mathbf{p}(y) \le v^h\}dy,$$

where

$$S_m(x) = \gamma + \int_{y \in (x,1)} \sigma(y) \left[\mathbf{q}(y) \mathbb{1}\{\mathbf{p}(y) \le v^h\} + (1 - \mathbf{q}(y)) \mathbb{1}\{\mathbf{p}(y) \le v^h\} \right] dy, \forall x \in (0,1)$$

Note that we can replace $\gamma \mathbf{q}(1-)$ with $\gamma \mathbf{q}(1)$ in the above: if $\gamma > 0$, then $\mathbf{q}(\cdot)$ can only be sustained if it is continuous at 1 by Lemma 3.

If we take $\gamma = \mu(1)$, the two systems are equivalent. For every $x \in (0, 1)$, the downstream sales at location x in m coincide with the mass of goods having vintage x in w.

H Omitted Proofs for Section 6.1

Proof of Proposition 4. Fix some sorting equilibrium $m = (\mathbf{p}, D, \mathbf{q}, \gamma)$. By Proposition 7, it is a \hat{x} -threshold market outcome for some outlet threshold \hat{x} . Clearly, m is suboptimal if it has zero sales. As we are allowing for disposal, non-zero sales no longer imply that some consumers must visit outlets. We must consider two cases.

Case 1: Outlets are visited. Suppose that the sales are positive and some consumers shop at outlet locations $\int_{y \in [\hat{x},1)} dD(y) > 0$. By Lemma 7, the seller's payoff is:

$$\begin{split} V^{S}(\mathbf{p}, D, \mathbf{q}, \gamma) &= TS(\mathbf{p}, D, \mathbf{q}, \gamma) - \mathbf{q}(\hat{x})(v^{h} - v^{l}) \\ &= \left(\int_{y \in [\hat{x}, 1)} dD(y) + \gamma \right) (1 - \mathbf{q}(\hat{x})) \left(\frac{\pi}{1 - \pi} v^{h} + v^{l} \right) \\ &- (1 + \gamma) \mathbf{q}(\hat{x})(v^{h} - v^{l}) - \gamma(\kappa + v^{l}). \end{split}$$

For a fixed outlet consumer share $\int_{y \in [\hat{x},1)} dD(y)$, seller's payoff is decreasing in $\mathbf{q}(\hat{x})$. If D has finitely many visited non-outlet locations, then $\mathbf{q}(\hat{x}) > \lambda\left(\int_{y \in [\hat{x},1)} dD(y)\right)$ by Lemma 9.

But similar to Lemma 10, we can construct a market outcome with a uniform shopping strategy that will result in the quality composition exactly $\lambda\left(\int_{y\in[\hat{x},1)} dD(y)\right)$ at the outlet threshold \hat{x} while preserving the share of outlets to be $\int_{y\in[\hat{x},1)} dD(y)$. Hence, the seller can improve upon her profit by deviating to this other sorting equilibrium with no atoms in the consumer strategy.

Case 2: Outlets are not visited. If $\int_{y \in [\hat{x},1]} dD(y) = 0$, then the seller may extract the whole total surplus with a constant price v^h :

$$V^{S}(\mathbf{p}, D, \mathbf{q}, \gamma) \leq TS(\mathbf{p}, D, \mathbf{q}, \gamma)$$

= $\gamma(1 - \mathbf{q}(\hat{x})) \left(\frac{\pi}{1 - \pi}v^{h} + v^{l}\right) - \gamma \mathbf{q}(\hat{x})(v^{h} - v^{l}) - \gamma(\kappa + v^{l})$

As TS is decreasing in $\mathbf{q}(\hat{x})$, there is again a profitable deviation towards a sorting equilibrium with no atoms.

I Omitted Proofs for Section 6.3

In the model with heterogeneous consumers we may describe the product flows by letting:

$$D_m(x) = \int_{\mathbf{x}(\theta) \le x: \mathbf{p}(\mathbf{x}(\theta)) \le \theta} dF(\theta)$$

to be the effective mass of consumers drawing goods from locations [0, x] in a market outcome $(\mathbf{p}, \mathbf{x}, \mathbf{q})$. Then, \mathbf{q} is sustained by (\mathbf{p}, \mathbf{x}) in the model of heterogenous consumers if and only if it is sustained in the baseline model by (\mathbf{p}, D_m) .

And a market outcome $(\mathbf{p}, \mathbf{x}, \mathbf{q})$ is a sorting equilibrium if $(i) \mathbf{q}$ is sustained by (\mathbf{p}, \mathbf{x}) and $(ii) \mathbf{x}$ is IC given (\mathbf{p}, \mathbf{q}) .

Recall that for every market outcome $m = (\mathbf{p}, \mathbf{x}, \mathbf{q}), Q_m(\theta)$ is the induced allocation of quality for type θ : $Q_m(\theta) = \mathbf{q}(\mathbf{x}(\theta))$. Let U_m be the induced consumer payoff: $U_m(\theta) = V^B(m|\theta)$.

Proof of Proposition 5. Neither of the results in Appendix B rely on $D_m(1) = 1$ (rather than any smaller). In particular, from Lemma 5 in any market outcome with positive sales, there is a positive mass of consumers visiting outlet locations. If the outlet threshold is

$$\hat{x} = \inf\{x : \mathbf{p}(x) \le v^l\}$$

, then from Lemma 6, it holds a positive share of high-quality goods $\mathbf{q}(\hat{x}) > 0$. And from Proposition 7, all locations in $(\hat{x}, 1)$ are outlet locations D-(a.s.). In addition, almost all such locations hold the same quality composition of $\mathbf{q}(\hat{x})$.

Lemma 12. In every sorting equilibrium $m = (\mathbf{p}, \mathbf{x}, \mathbf{q})$ with positive sales:

- (i) Q_m is increasing
- (ii) and every $\theta > v^l$:

$$U_m(\theta) = U_m(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} Q_m(s) ds,$$

where

$$\hat{\theta} = \sup\{\theta : \mathbf{p}(\mathbf{x}(\theta)) \le v^l\}.$$

Proof. As $\mathbf{q}(\hat{x}) > 0$, and there is a positive measure of outlet locations in $(\hat{x}, 1)$, then any type $\theta > v^l$ receives a strictly positive payoff, and $\mathbf{p}(\mathbf{x}(\theta)) < \theta$. Similar to the proof in Proposition 7, otherwise a consumer of type $\theta > v^l$ may deviate to a location in $(\hat{x}, 1)$ that has low price of at most v^l and offers a positive probability of finding a high-quality good.

Then, from (IC), for any $\theta, \theta' > v^l$, θ does not have a profitable deviation towards $\mathbf{x}(\theta')$ if and only if:

$$U_m(\theta) \ge U_m(\theta') + Q_m(\theta')(\theta - \theta').$$

Note that for any market outcome with positive total sales, $\hat{\theta} > v^l$, to ensure a positive measure of outlet shoppers. Then, using the standard argument, we obtain that Q_m agrees with IC only if Q_m is increasing on $[\hat{\theta}, v^h]$ and consumer's equilibrium payoff satisfies the envelope formula:

$$U_m(\theta) = U_m(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} Q_m(s) ds$$

Lemma 13. In every sorting equilibrium with positive sales:

- (i) if $\theta < \hat{\theta}$, then $\mathbf{x}(\theta) \ge \hat{x}$
- (ii) **x** is decreasing on $(\hat{\theta}, v^h)$

Proof. **Part** (*i*). Suppose not, and there exists $\tilde{\theta} < \hat{\theta}$, such that $\mathbf{x}(\theta) < \hat{x}$. By definition of \hat{x} , then $\tilde{\theta}$ visits a non-outlet location $\mathbf{p}(\mathbf{x}(\tilde{\theta})) > v^l$.

By definition of $\hat{\theta}$, either $\hat{\theta}$ visits an outlet location with quality composition quality composition $\mathbf{q}(\hat{x})$, or there exists θ' arbitrarily close to $\hat{\theta}$ visiting such location. As $\mathbf{p}(\mathbf{x}(\tilde{\theta})) > v^l$, then $\tilde{\theta}$ does not have a profitable deviation to one of the outlet locations only if he visits a location with a better average quality $Q_m(\tilde{\theta}) > \mathbf{q}(\hat{x})$. But then we obtain a contradiction with monotonicity of Q_m from Lemma 12.

Part (*ii*). By definition of $\hat{\theta}$, all consumer types above $\hat{\theta}$, shop at non-outlet locations contained in $(0, \hat{x})$. Suppose by a way of contradiction that there exist $\theta_1 > \theta_2 > \hat{\theta}$, such that $\hat{x} > \mathbf{x}(\theta_1) > \mathbf{x}(\theta_2)$. By Lemma 4, $(1 - \mathbf{q}(\mathbf{x}(\theta_1)))S_m(\mathbf{x}(\theta_1)) = (1 - \mathbf{q}(\mathbf{x}(\theta_2)))S_m(\mathbf{x}(\theta_2))$. Hence, to satisfy the monotonicity condition for Q_m , it must be that a zero mass of consumers shop at location in $[\mathbf{x}(\theta_2), \mathbf{x}(\theta_2)]$. This is only possible if a non-empty subset of consumer types in (θ_2, θ_1) visits either locations $(0, \mathbf{x}(\theta_1))$, or locations $(\mathbf{x}(\theta_2), 1)$. Either way, we may find some consumer types types $\theta'_1 > \theta'_2$, for whom $\mathbf{x}(\theta'_1) > \mathbf{x}(\theta'_2)$ and there is a non-trivial mass of consumers shopping between their visited locations $(\mathbf{x}(\theta'_2), \mathbf{x}(\theta'_1))$. Then, by Lemma 4, $Q_m(\theta'_1) < Q_m(\theta'_2)$ violating monotonicity of Q_m .

Proposition 5 now follows from Lemma 13.

Lemma 14. Suppose $m = (\mathbf{p}, \mathbf{x}, \mathbf{q})$ is a sorting equilibrium. For a given threshold outlet shopper $\hat{\theta}$, the induced allocation $Q_m(\cdot) = Q^{\hat{\theta}}(\cdot)$ on $[\hat{\theta}, v^h]$, where:

$$\frac{Q^{\hat{\theta}}(\theta)}{1-Q^{\hat{\theta}}(\theta)} = W\left(\frac{\pi}{1-\pi} \exp\left[\frac{\pi}{1-\pi} - \frac{1-F(\theta)}{F\left(\hat{\theta}\right)\left(1-Q^{\hat{\theta}}\left(\hat{\theta}\right)\right)}\right]\right)$$

and $Q^{\hat{\theta}}\left(\hat{\theta}\right)$ is satisfies:

$$\ln\left(\frac{\pi}{1-\pi}\frac{1-Q^{\hat{\theta}}\left(\hat{\theta}\right)}{Q^{\hat{\theta}}\left(\hat{\theta}\right)}\right) = \frac{1}{F\left(\hat{\theta}\right)\left(1-Q^{\hat{\theta}}\left(\hat{\theta}\right)\right)} - \frac{1}{1-\pi}$$

Proof. The proof for the shape $Q^{\hat{\theta}}(\theta)$ on $(\hat{\theta}, v^h]$ is analogous to Lemma 8. The expression for $Q^{\hat{\theta}}(\hat{\theta})$ follows from continuity of $\mathbf{q}(\cdot)$ at $\mathbf{x}(\hat{\theta})$ due to Lemma 3.

Proof of Proposition 6. If Direction. From the outlet shoppers, the seller collects revenue of at most v^l . From non-outlet shoppers, the seller's revenue can be computed as the difference between the total surplus and the consumers' expected payoff:

$$V^{S}(\mathbf{p}, \mathbf{x}, \mathbf{q}) \leq \int_{\hat{\theta}}^{v^{h}} (\theta Q_{m}(\theta) - U_{m}(\theta)) dF(\theta) + F(\hat{\theta}) v^{l}$$

Due to the envelope condition for the induced consumer payoff in Lemma 12, the seller's payoff is at most:

$$V^{S}(\mathbf{p}, \mathbf{x}, \mathbf{q}) \leq \int_{\hat{\theta}}^{v^{h}} \theta Q_{m}(\theta) - \left(\int_{\hat{\theta}}^{\theta} Q_{m}(s) ds\right) dF(\theta) + F(\hat{\theta})v^{l} - U_{m}(\hat{\theta})(1 - F(\hat{\theta}))$$

Applying integration by parts,

$$-\int_{\hat{\theta}}^{v^{h}} \left(\int_{\hat{\theta}}^{\theta} Q_{m}(s)ds\right) dF(\theta) = \int_{\hat{\theta}}^{v^{h}} \left(\int_{\hat{\theta}}^{\theta} Q_{m}(s)ds\right) d(1-F(\theta))$$

$$= 0 - 0 - \int_{\hat{\theta}}^{v^h} (1 - F(\theta)) Q_m(\theta) d\theta$$

So that we can rewrite the boundary on the seller's payoff as:

$$V^{S}(\mathbf{p}, \mathbf{x}, \mathbf{q}) \leq \int_{\hat{\theta}}^{v^{h}} Q_{m}(\theta) \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) dF(\theta) + F(\hat{\theta})v^{l} - U_{m}(\hat{\theta})(1 - F(\hat{\theta}))$$

The threshold outlet shopper $\hat{\theta}$ gets a payoff at least $Q_m(\hat{\theta})(\hat{\theta} - v^l)$ (as he shops at or in the neighborhood of an outlet location):

$$V^{S}(\mathbf{p}, \mathbf{x}, \mathbf{q}) \leq \int_{\hat{\theta}}^{v^{h}} Q_{m}(\theta) \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) dF(\theta) + F(\hat{\theta})v^{l} - Q_{m}(\hat{\theta})(\hat{\theta} - v^{l})(1 - F(\hat{\theta}))$$

as the threshold outlet shopper gets a payoff at least $Q_m(\hat{\theta})(\hat{\theta} - v^l)$ (as he shops at or in the neighborhood of an outlet location). Finally, since $Q_m(\cdot) = Q^{\hat{x}}(\cdot)$ on $[\hat{\theta}, v^h]$ by Lemma 14, we get that the seller's payoff is at most:

$$V^{S}(\mathbf{p}, \mathbf{x}, \mathbf{q}) \leq \int_{\hat{\theta}}^{v^{h}} Q^{\hat{\theta}}(\theta) \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) dF(\theta) + F(\hat{\theta})v^{l} - Q^{\hat{\theta}}(\hat{\theta})(\hat{\theta} - v^{l})(1 - F(\hat{\theta})),$$

as required. Finally, the seller can achieve this bound for any $\hat{\theta}$ by shifting all prices up, so that all outlets have a price of v^l .

Only if direction. Let me construct a sorting equilibrium for a given outlet shopper $\hat{\theta} \in (v^l, v^h]$. Let consumer strategy be defined as:

$$\mathbf{x}(\theta) = \frac{1}{3} \left(1 + \frac{v^h - \theta}{v^h - v^l} \right)$$

Let the quality composition be specified as:

$$\mathbf{q}(x) = \begin{cases} \pi, \text{ if } x \le 1/3\\ Q^{\hat{\theta}}(\mathbf{x}^{-1}(x)), \text{ if } x \in [1/3, \mathbf{x}\left(\hat{\theta}\right)]\\ Q^{\hat{\theta}}\left(\hat{\theta}\right), \text{ if } x \ge \mathbf{x}\left(\hat{\theta}\right) \end{cases}$$

It is sustained by (\mathbf{p}, \mathbf{x}) , by construction (see Lemma 14).

For every location $x \in [\mathbf{x}(\hat{\theta}), 1)$, set the price at v^l : $\mathbf{p}(x) = v^l$. For all visited remaining
locations, specify the prices from the envelope condition on the consumer payoff $U_m(\theta)$:³³

$$\begin{aligned} \forall \theta > \hat{\theta} : \ \mathbf{p}(\mathbf{x}(\theta)) &= \theta - \frac{U_m(\theta)}{Q^{\hat{\theta}}(\theta)} \\ &= \theta - \frac{U_m\left(\hat{\theta}\right)}{Q^{\hat{\theta}}(\theta)} - \int_{\hat{\theta}}^{\theta} \frac{Q^{\hat{\theta}}(s)}{Q^{\hat{\theta}}(\theta)} ds \\ &= \theta - \frac{Q^{\hat{\theta}}\left(\hat{\theta}\right)}{Q^{\hat{\theta}}(\theta)} (\hat{\theta} - v^l) - \int_{\hat{\theta}}^{\theta} \frac{Q^{\hat{\theta}}(s)}{Q^{\hat{\theta}}(\theta)} ds \end{aligned}$$

For all locations that are upstream of 1/3 set prices to be prohibitively high, e.g. $\mathbf{p}(x) = v^h$ for all $x < \frac{1}{3}$.

By construction, none of the consumer types has a profitable devotion, and $(\mathbf{p}, \mathbf{x}, \mathbf{q})$ is IC. By Lemma 8, \mathbf{q} is sustained by (\mathbf{p}, \mathbf{x}) on $[0, \mathbf{x}(\hat{\theta})]$, and \mathbf{q} is sustained on $[\mathbf{x}(\hat{\theta}), 1]$ by Lemma 4 (*ii*).

³³Every type pays the price only if he finds a high-quality good.

A Online Appendix

This section includes additional results omitted in the main Appendix of the paper. Appendix OA1 formally describes the two-store model, which was an illustration in the Introduction of the paper.

Appendix OA2 constructs the sorting equilibrium for the model with multiple qualities.

OA1 Two-Store Model

This section formalizes a two-store model, which was used as an illustration in the Introduction of the paper. It derives the key comparative statics results summarized in Figure 1 in the Introduction.

There are two stores: a flagship and an outlet. Each store holds a continuum of products of mass 1. The goods have binary quality as in Section 3. The share of high-quality goods at store $i \in \{f, o\}$ is denoted as q^i . Time is discrete and runs over an infinite horizon, $t \in \{1, 2, ...\}$. Each period, a mass $\lambda \in (0, 1)$ of short-lived consumers arrives at the market. Flagship charges a full price $p^f > v^l$, whereas the outlet sells goods at a markdown $p^o = v^l$. **Consumer Behavior**. Upon visiting the store, each consumer is matched to a single product at random. The probability of getting matched to a high-quality good at location i is q^i . Each product is matched to at most one consumer. Consumers choose between the two stores given prices and shares of high-quality goods. Let σ denote the share of consumers who choose the flagship store. Consumers select their shopping strategy σ to maximize the expected payoff given the prices and the quality composition:

$$V^{B}(p^{f}, \sigma, q^{f}, q^{o}) = \sigma q^{f}(v^{h} - p^{f}) + (1 - \sigma)q^{o}(v^{h} - v^{l}).$$

Quality Composition Evolution. The stock reallocation is a discrete analog of the continuous model. Inventory is reallocated downstream from production to the flagship to the outlet, to maintain both stores at their full capacity.

Suppose at the beginning of period t, the proportion of high-quality goods at each store $i \in \{f, o\}$ is given by q_t^i . Consider the outlet first. At the outlet, consumers purchase any product type they find. Therefore, total sales at the outlet in any given period equal its consumer flow $(1 - \sigma)\lambda$, with a share q_t^o of these sales being of high quality. To replenish the outlet, the seller ships inventory from the flagship equal to the total outlet sales, $\lambda(1 - \sigma)$. The share of high-quality goods in the shipments is the proportion of high-quality goods in the flagship's after-sales remaining inventory, denoted $q_{t,a}^f$. Thus, the total change in the

mass of high-quality items³⁴ at the outlet is

$$\Delta q_t^o = q_{t,a}^f \lambda (1-\sigma) - q_t^o \lambda (1-\sigma).$$

Next, consider the evolution of the quality composition at the flagship store. Consumers only purchase high-quality goods there. The total mass of purchases at the flagship equals $\lambda \sigma q_t^f$, the mass of consumers who find a high-quality product. The total remaining stock after purchases is then $1 - \lambda \sigma q_t^f$, while the remaining mass of high-quality goods is $q_t^f(1 - \lambda \sigma)$. The resulting after-sales proportion of high-quality, $q_{t,a}^f$ is equal to

$$q_{t,a}^f = q_t^f \frac{(1 - \lambda \sigma)}{1 - \lambda \sigma q_t^f}.$$

The flagship gets restocked to full capacity once consumer purchases and shipments to the outlet are complete. The total mass of new inventory ordered from the production plant to the flagship equals the mass of total sales at both stores in period t, which is $q_t^f \lambda \sigma + \lambda(1-\sigma)$. A fraction π of these new items is of high quality. Hence, the change in the flagship's share of high-quality items is given by

$$\Delta q_t^f = \pi (q_t^f \lambda \sigma + \lambda (1 - \sigma)) - \lambda \sigma q_t^f - q_{t,a}^f \lambda (1 - \sigma).$$

Sorting Equilibrium. The quality composition (q^f, q^o) is sustained by consumer strategy σ^{35} when the proportion of high-quality goods at both stores remains constant over time: $\Delta q_t^i = 0$ for $q_t^i = q^i$.

Flagship price p^f , consumer strategy σ , and the quality composition (q^f, q^o) form a sorting if (i) (q^f, q^o) is sustained by consumer strategy σ and (ii) σ is consumer-optimal given (p^f, q^f, q^o) .

OA1.1 Sorting Equilibria in a Two-Store Model

First, I fix the consumer strategy σ and analyze the quality composition it sustains. Lemma 15 shows that each consumer st

Lemma 15. If $\sigma = 1$, then any $(q^f, q^o) \in [0, 1]^2$, such that $q^f = 0$ is sustained by σ . For every $\sigma < 1$, there exists a unique quality composition, $(\mathbf{q}^f(\sigma), \mathbf{q}^o(\sigma))$ it sustains. Moreover,

³⁴Given the stock of either store is normalized to one, the change in the proportion of high-quality goods at any store coincides with the change of their mass.

 $^{^{35}\}mathrm{We}$ can skip the prices from the definition, as we already fixed them above v^l for the flagship and at v^l for the outlet.

- (i) $\mathbf{q}^f(\sigma) > \mathbf{q}^o(\sigma) > 0$
- (ii) both $\mathbf{q}^{f}(\cdot)$ and $\mathbf{q}^{o}(\cdot)$ are decreasing, with $\mathbf{q}^{f}(0) = \mathbf{q}^{o}(0) = \pi$,
- (iii) and $\mathbf{q}^f(\cdot)/\mathbf{q}^o(\cdot)$ is strictly increasing.

Proof. By definition, (q^f, q^o) is sustained by σ whenever:

$$\lambda(1-\sigma)\frac{(1-\lambda\sigma)q^f}{1-\lambda\sigma q^f} - \lambda(1-\sigma)q^o = 0, \tag{14}$$

$$\pi(q^f \lambda \sigma + \lambda(1 - \sigma)) - \lambda \sigma q^f - \frac{(1 - \lambda \sigma)q^f}{1 - \lambda \sigma q^f} \lambda(1 - \sigma) = 0.$$
⁽¹⁵⁾

Step 1: $\sigma = 1$. In this case, Equation (14) is satisfied for any q^o , q^f . And Equation (14) at $\sigma = 1$ becomes:

$$\pi q^f \lambda - \lambda q^f = 0.$$

As $\pi < 1$, the above is only satisfied when $q^f = 0$. Step 2: $\sigma < 1$. In this case, Equation (14) reduces to:

$$q^o = \frac{(1 - \lambda \sigma)q^f}{1 - \lambda \sigma q^f}$$

That is, the outlet's quality composition coincides with the flagship's after-sales average quality. In addition, if (q^f, q^o) is sustained by σ , then $q^o < q^f$.

To solve for q^f which can be sustained by σ , define $\Psi(q^f, \sigma)$:

$$\Psi(\sigma, q^f) = \pi(\sigma q^f + (1 - \sigma)) - q^f \sigma - \delta q^f (1/\lambda - \sigma) - \frac{q^f (1 - \lambda \sigma)(1 - \delta)}{1 - q^f \lambda \sigma} (1 - \sigma)$$

Then, q^f is sustained by σ whenever $\Psi(q^f, \sigma) = 0$. I now show that there exists a unique such q^f for every σ . To that end, it is sufficient to show that $\Psi(\cdot, \sigma)$ is decreasing in q^f for every σ and $\Psi(\cdot, \sigma)$ changes its sign at some interior q^f .

Existence. $\Psi(0, \sigma) = \pi(1 - \sigma) \ge 0$, where the inequality is strict if and only if $\sigma < 1$. On the other hand, $\Psi(\pi, \sigma)$ is:

$$\Psi(\pi,\sigma) = \pi(1-\sigma) - \sigma(1-\pi)\pi - (1-\sigma)\frac{\pi(1-\lambda\sigma)}{1-\pi\sigma\lambda}$$
$$= \frac{\pi(1-\pi)\lambda\sigma(1-\sigma)}{1-\pi\sigma\lambda} - \sigma(1-\pi)\pi$$
$$= -\pi(1-\pi)\sigma\frac{\lambda\sigma(1-\pi) + (1-\lambda)}{1-\pi\sigma\lambda} \le 0$$

Hence, for every $\sigma < 1$ there exist $q^f \in [0, 1]$, such that $\Psi(q^f, \sigma) = 0$ (by the Intermediate Value Theorem due to continuity of Ψ in q^f).

Uniqueness. Now, let me verify that $\Psi(q^f, \sigma)$ is decreasing in q^f :

$$\partial_{q^f} \Psi(q^f, \sigma) = -\sigma(1 - \pi) - \frac{(1 - \sigma)(1 - \lambda\sigma)}{(1 - q^f \sigma \lambda)^2} < 0$$

Hence, an intersection with 0 is unique for every σ . I can denote such an intersection as $\mathbf{q}^{f}(\sigma)$. Given the uniqueness of the flagship quality composition that can be sustained by σ , a sustained q^{o} is also unique. To summarize, for every $\sigma < 1$, there is a unique quality composition $(\mathbf{q}^{f}(\sigma), \mathbf{q}^{o}(\sigma))$ it sustains. In addition, $\mathbf{q}^{f}(\sigma) > \mathbf{q}^{o}(\sigma) > 0$, for ever $\sigma < 1$, and And $\mathbf{q}^{f}(0) = \mathbf{q}^{o}(0) = \pi$.

Step 3: comparative statics of $\mathbf{q}^{f}(\sigma)$. By an Implicit Function Theorem, we have:

$$\partial_{\sigma} \mathbf{q}^{f}(\sigma) = -\partial_{\sigma} \Psi(\mathbf{q}^{f}(\sigma), \sigma) / \partial_{q^{f}} \Psi(\mathbf{q}^{f}(\sigma), \sigma)$$

In Step 2, we showed $\partial_{q^f} \Psi(\mathbf{q}^f(\sigma), \sigma) < 0$. Then, the sign of $\partial_{\sigma} \mathbf{q}^f(\sigma)$ is determined by the sign of $\partial_{\sigma} \Psi(\mathbf{q}^f(\sigma), \sigma)$. I now show that $\partial_{\sigma} \Psi(q^f, \sigma) < 0$ for every $q^f \leq \pi$.

$$\begin{aligned} \partial_{\sigma}\Psi(q^{f},\sigma) &= -\pi - q^{f}(1-\pi) + (1+\lambda-2\lambda\sigma)\frac{q^{f}}{1-\sigma q^{f}\lambda} \\ &-\lambda(1-\sigma)(1-\lambda\sigma)\left(\frac{q^{f}}{1-\sigma q^{f}\lambda}\right)^{2} \\ &= -\pi - q^{f}(1-\pi) + 2\frac{(1-\lambda\sigma)q^{f}}{1-\sigma q^{f}\lambda} - \left(\frac{(1-\lambda\sigma)q^{f}}{1-\sigma q^{f}\lambda}\right)^{2} \\ &- (1-\lambda)\frac{q^{f}}{1-\sigma q^{f}\lambda} + (1-\lambda)(1-\lambda\sigma)\left(\frac{q^{f}}{1-\sigma q^{f}\lambda}\right)^{2} \end{aligned}$$

Using

$$- (1-\lambda)\frac{q^f}{1-\sigma q^f \lambda} + (1-\delta)(1-\lambda)(1-\lambda\sigma)\left(\frac{q^f}{1-\sigma q^f \lambda}\right)^2$$
$$= (1-\lambda)q^f \left(\frac{1}{1-\sigma q^f \lambda}\right)^2 \left[-(1-q^f \sigma \lambda) + q^f(1-\lambda\sigma)\right] \le 0$$

we can bound $\partial_{\sigma} \Psi(q^f, \sigma)$ by:

$$\partial_{\sigma}\Psi(q^{f},\sigma) \leq -\pi - q^{f}(1-\pi) + 2\frac{(1-\lambda\sigma)q^{f}}{1-\sigma q^{f}\lambda} - \left(\frac{(1-\lambda\sigma)q^{f}}{1-\sigma q^{f}\lambda}\right)^{2}.$$

Additionally, as $\frac{(1-\lambda\sigma)q^f}{1-\sigma\lambda q^f} \leq q^f$ and $2x - x^2$ increasing in x for $x \leq 1$, we can further bound the above:

$$\partial_{\sigma}\Psi(q^{f},\sigma) \leq -\pi - q^{f}(1-\pi-\delta) + 2q^{f} - \left(q^{f}\right)^{2}$$

$$\tag{16}$$

Now, I show that the Expression (16) that bounds $\partial_{\sigma} \Psi(q^f, \sigma)$ is increasing in q^f . Differentiating it with respect to q^f , we get:

$$\partial_{q^{f}} \left(-\pi - q^{f} (1 - \pi) + 2q^{f} - (q^{f})^{2} \right) = 1 + \pi - 2(1 - \delta)q^{f}$$
$$\geq 1 + \pi - 2\pi$$
$$= (1 - \pi) > 0$$

Hence, we can bound $\partial_{\sigma} \Psi(q^f, \sigma)$ by plugging $q^f = \pi$ in the Expression (16):

$$\partial_{\sigma} \Psi(q^f, \sigma) \leq -\pi - \pi (1 - \pi) + 2\pi - \pi^2$$

 $\leq -\pi - \pi (1 - \pi) + 2\pi - \pi^2 = 0.$

Then, $\partial_{\sigma} \Psi(q^f, \sigma) \leq 0$ and $\partial_{\sigma} \Psi(q^f, \sigma) < 0$ for $q^f < \pi$.

Step 4: comparative statics for the sorting precision $\mathbf{q}^{f}(\sigma)/\mathbf{q}^{o}(\sigma)$.

Differentiating the sorting precision with respect to σ , we get:

$$\partial_{\sigma} \mathbf{q}^{f}(\sigma)/\mathbf{q}^{o}(\sigma) = \partial_{\sigma} \frac{1 - \lambda \sigma \mathbf{q}^{f}(\sigma)}{1 - \lambda \sigma} = \frac{-\lambda \mathbf{q}^{f}(\sigma)(1 - \lambda \sigma) + \lambda(1 - \lambda \sigma \mathbf{q}^{f}(\sigma))}{(1 - \lambda \sigma)^{2}}$$
$$= \frac{\lambda(1 - \mathbf{q}^{s}(\sigma)) - \lambda \sigma \partial_{\sigma} \mathbf{q}^{f}(\sigma)}{(1 - \lambda \sigma)^{2}} > 0$$

where the we use $\partial_{\sigma} \mathbf{q}^{f}(\sigma) < 0$ by Step 3. In addition, as $\mathbf{q}^{f}(\cdot)$ is decreasing but $\mathbf{q}^{f}(\cdot)/\mathbf{q}^{o}(\cdot)$ is increasing, then $\mathbf{q}^{o}(\cdot)$ is decreasing. This completes the proof of the lemma.

Convergence to the Steady State. I now discuss the convergence of the quality composition to the sustained steady state to interpret the sorting equilibrium.

The quality composition evolution described above assumes a constant consumer strategy. To interpret, suppose consumers do not see the time of their arrival or the current quality composition. In equilibrium, they correctly anticipate the long-run quality composition at the two stores, which dominates their beliefs.

To make sure this interpretation is valid, we must verify that for a given σ , the quality composition converges to a unique steady state. I illustrate convergence to the steady state in Figure 8. To establish convergence, note that Δq_t^f only depends on q_t^f and is decreasing in it. Hence, whenever $q_t^f > (<)\mathbf{q}^f(\sigma)$, $\Delta q_t^f < (>)0$. The quality composition at the flagship is pushed towards the steady state. In turn, for a given q_t^f , Δq_t^o is decreasing in q_t^o . Hence, the steady state is stable.



Figure 8: Phase Diagram

Interior Equilibrium. We can now formulate the main result for the two-store model: comparative statics of the interior sorting equilibrium with respect to the flagship price p^f .

Proposition 9. For every flagship price $p^f \in (v^l, v^h)$, there exists at most one interior sorting equilibrium (p^f, σ, q^f, q^o) . Moreover, in this equilibrium, if p^f increases

- (i) the customer share of the flagship store σ rises,
- (ii) the quality composition at both stores (q^f, q^o) gets worse,
- (iii) the sorting precision q^f/q^o rises,
- (iv) and the total steady-state per-period sales $\lambda[\sigma q^f + (1 \sigma)]$ decrease.

Proof. By Lemma 15, for every interior σ , there exists a unique quality composition that it sustains. In addition, at any such equilibrium, consumers must be indifferent between the two stores, which implies:

$$\frac{\mathbf{q}^f(\sigma)}{\mathbf{q}^o(\sigma)} = \frac{v^h - v^l}{v^h - p^f} \tag{IND}$$

By Lemma 15, the sorting precision $\frac{\mathbf{q}^{f}(\sigma)}{\mathbf{q}^{o}(\sigma)}$ is strictly increasing in σ . Then, for every price p^{f} , there exists at most one σ , where Equation (IND) is true. Consequently, by Lemma 15,

for every p^f there exists at most one interior sorting equilibrium. Moreover, as p^f becomes larger, the sorting precision must rise, which requires σ to increase in the interior equilibrium. Part (i) follows. Parts (ii), (iii) then follow from Lemma 15. Finally, the steady-state sales at this equilibrium are given by

$$\lambda[\sigma \mathbf{q}^f(\sigma) + (1-\sigma)]$$

and are decreasing in σ .

OA2 Sorting Equilibrium Construction for Multiple Qualities

For any sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$, and define a_m be the lowest-alwayspurchased quality:

$$a_m \equiv \min_{n \ge i \ge 1} \left\{ i : \int_{y: \mathbf{p}(y) > v^i} \sigma(y) dy = 0 \right\}$$

and define e_m be the lowest-ever-purchased quality:

$$e_m \equiv \min_{n \ge i \ge 1} \left\{ i : \int_{y: \mathbf{p}(y) \le v^i} \sigma(y) dy > 0 \right\}$$

Proposition 10. Consider two sorting equilibria $m_1, m_2 \in \mathcal{E}$ with positive sales and the same disposal rate $\gamma \geq 0$. If they induce the same lowest-ever-purchased quality, i.e. $e_{m_1} = e_{m_2}$, and the same consumer surplus, i.e. $V^B(m_1) = V^B(m_2)$, then they also induce the sale seller profit, i.e. $V^S(m_1) = V^S(m_2)$.

Proof. To prove Proposition 10, I characterize the construction of the sorting equilibrium with multiple qualities. First, I show that the choice of the induced consumer payoff pins down the lowest-always-purchased quality. Effectively, the seller picks the highest price through her choice of the consumer surplus.

Lemma 16. If a sorting equilibrium m induces consumer payoff $CS \ge 0$ and has the lowestpurchased-quality a_m , then:

$$\sum_{k \ge a^m} \pi(k)(v^k - v^{a_m - 1}) \ge CS \ge \sum_{k \ge a^m} \pi(k)(v^k - v^{a_m}),$$

Proof. Find the earliest visited location

$$\bar{y} = \sup\left\{y : \int_{x \in (0,y)} \sigma(x) dx = 0\right\}.$$

Then, it must be that in any right neighborhood of \bar{y} , consumers shop with positive probability, and we can find a converging sequence of visited locations $\{z_k\}$ with $z_k \to \bar{y}$, such that the consumer's payoff is given by :

$$CS = \sum_{i=1}^{n} \mathbf{q}(i|z_k)(v^i - \mathbf{p}(z_k))_+$$

 a_m is the lowest-always-purchased quality, then $\hat{x}_{a_m-1} > \bar{y}$ and by Proposition 7, $\mathbf{p}(\cdot) \in (v^{j+1}, v^j)$ $[\bar{y}, \hat{x}_{a_m-1}]$, *D*-a.s.. Then, along the sequence $\{z_k\}$, consumer payoff is bounded by:

$$\sum_{i=1}^{n} \mathbf{q}(i|z_k) \left(v^i - v^{a^m} \right)_+ \le CS \le \sum_{i=1}^{n} \mathbf{q}(i|z_k) \left(v^i - v^{a^m - 1} \right)_+$$

By Lemma 3, $\mathbf{q}(\cdot)$ is continuous at \bar{y} , then as $z_k \to \bar{y}_j$, $\mathbf{q}(i|z_k) \to \mathbf{q}(i|\bar{y})$. As no locations are visited on $(0, \bar{y})$, they all are outlets *D*-a.s.. Then, by Lemma 4 (*ii*): $\mathbf{q}(i|\bar{y}) = \mathbf{q}(i|0) = \pi(i)$, and we obtain:

$$\sum_{k \ge a^m} \pi(k) \left(v^k - v^{a^m} \right) \le CS \le \sum_{k \ge a^m} \pi(k) \left(v^k - v^{a^m - 1} \right)$$

as required.

Second, I characterize jumps that the purchase probabilities make when the price crosses the product's potential values.

Lemma 17. Consider a sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ with positive total sales and consumer surplus $CS \ge 0$. For any $i \in \{a_m - 1, e_m\}$, the purchase probability at the threshold location $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \le v^i\}$, jumps by:

$$\rho_m(\hat{x}_i-) + (1-\rho_m(\hat{x}^i-))\frac{\pi(i)}{\sum_{l \le i} \pi(l)}.$$

Moreover, the expected value of the product of a product conditional on purchase is constant between any two threshold locations $[\hat{x}_i, \hat{x}_{i-1})$ and must satisfy:

$$E_m^i = \frac{\rho_m(\hat{x}_i) - E_m^{i+1} + (1 - \rho_m(\hat{x}_i)) \frac{\pi(i)}{\sum_{l \le i} \pi(l)} v^i}{\rho_m(\hat{x}_i)}$$

Proof. At any threshold \hat{x}_i , the probability of purchase jumps upwards by exactly $\mathbf{q}(i|\hat{x}_i)$, as consumers start purchasing quality *i*. Given Lemma 4 (*ii*), there is no learning about quality *i* relative to any other quality $(0, \hat{x}_i)$ which is not purchased on this interval:

$$\frac{\mathbf{q}(i|\hat{x}_i)}{\sum_{l\leq i}\mathbf{q}(l|\hat{x}_i)} = \frac{\mathbf{q}(i|\hat{x}_i)}{1-\rho_m(\hat{x}^i-)} = \frac{\pi(i)}{\sum_{l\leq i}\pi(l)}.$$

Then, we can jump in the purchasing probability at \hat{x}_i must be:

$$\rho_m(\hat{x}_i) = \rho_m(\hat{x}_i) + \mathbf{q}(i|\hat{x}_i) = \rho_m(\hat{x}_i) + (1 - \rho_m(\hat{x}^i)) \frac{\pi(i)}{\sum_{l \le i} \pi(l)}.$$

The part about the conditional expected value is similarly implied by Lemma 4 (ii).

Using the consumer indifference condition between all visited locations, we can derive further restrictions on the equilibrium purchase probabilities at the threshold locations.

Lemma 18. Consider a sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ with positive total sales and consumer surplus $CS \ge 0$. For any $i \in \{a_m - 1, e_m\}$, the purchase probability at the threshold location $\hat{x}_i = \inf\{x \in X : \mathbf{p}(x) \le v^i\}$ satisfies:

$$\frac{CS}{\rho_m(\hat{x}_{i-1}-)} = \frac{CS}{\rho_m(\hat{x}_i)} + v^i - v^{i-1},$$

where $\rho_m(\hat{x}_i) = \rho_m(\hat{x}_i-) + (1 - \rho_m(\hat{x}_i-)) \frac{\pi(i)}{\sum_{l \le i} \pi(l)}$
and $\rho_m(\hat{x}_{a_m-1}-) = \frac{CS}{\frac{\sum_{l \ge a_m} \pi(l)v^l}{\sum_{l \ge a_m} \pi(l)} - v^{a_m-1}}$

Proof. Lemma 17 implies the following simple identity at any threshold location:

$$\rho(\hat{x}_{i+1})\left(E_m^{i+1} - v^{i+1}\right) = \rho(\hat{x}_{i+1} -)\left(E_m^{i+2} - v^{i+1}\right).$$
(17)

Step 1: *indifference condition*. I show that for every $i \leq a_m - 1$ such that $\hat{x}_i < 1$, we must have:

$$CS = \rho_m(\hat{x}_i)(E_m^{i-1} - v^i),$$
$$\int_{y \in [\hat{x}_i, \hat{x}_{i-1})} \sigma(y) dy > 0.$$

I establish the above by induction.

Initial Iteration: $i = a_m - 1$. If $\hat{x}_{a_m-1} < 1$, then the price is below v^{a_m-1} at some locations in any left neighborhood of \hat{x}_{a_m-1} . By definition of the lowest-always-purchased quality:

$$\int_{y\in [\hat{x}_{a_m}, \hat{x}_{a_m-1})} \sigma(y) dy > 0.$$

The quality composition is continuous by Lemma 3. Then, the consumer's payoff is at least:

$$V^{B}(\mathbf{p}, \sigma, \mathbf{q}, \gamma) = CS \ge \sum_{l \ge a^{m}} \mathbf{q}(l | \hat{x}_{a_{m}-1}) \left(v^{l} - v^{a_{m}-1} \right)$$
$$= \rho(\hat{x}_{a_{m}-1}) \left(E_{m}^{a_{m}} - v^{a_{m}-1} \right)$$

Consider the last visited location with the price above v^{a_m-1} :

$$\hat{y}_{a_m-1} = \sup_{x \in [0, \hat{x}_{a_m-1}]} \left\{ \int_{z \in (y, \hat{x}_{a_m-1}]} \sigma(y) dy = 0 \right\}.$$

Then, consumers do not visit locations in $[\hat{y}_{a_m-1}, \hat{x}^{j+1}]$ and $\rho(\hat{y}_{a_m-1}) = \rho(\hat{x}_{a_m-1}-)$. In addition, consumers shop with a positive probability in the right neighborhood of \hat{y}_{a_m-1} . If the consumer's shopping strategy is optimal, there is a sequence $\{z_l\}_{l=1}^{\infty}$ converging to \hat{y}_{a_m-1} , such that consumers obtain their payoff at each of locations $\{z_l\}_{l=1}^{\infty}$:

$$CS = \rho(z_l)(E_m^{a_m} - \mathbf{p}(z_l)) \le \rho(z_l) \left(E_m^{a_m} - v^{a_m - 1}\right) \xrightarrow[z_l \to \hat{y}_{a_m - 1}]{} \rho(\hat{x}_{a_m - 1} -) \left(E_m^{a_m} - v^{a_m - 1}\right)$$

Together, the two bounds imply that:

$$CS = \rho(\hat{x}_{a_m-1}) \left(E_m^{a_m} - v^{a_m-1} \right)$$

Iteration *i*. Suppose the statement is true for all $k \ge i + 1$, and let us verify that it must then be true for *i*. If $\hat{x}_i = 1$, then we are done. Otherwise, suppose that $\hat{x}_i < 1$, then the consumer payoff is at least:

$$V^B(\mathbf{p}, \sigma, \mathbf{q}, \gamma) = CS \ge \rho(\hat{x}_i -)(E_m^{i+1} - v^i)$$

If $\int_{y\in[\hat{x}_{i+1},\hat{x}_i]}^{\hat{x}_i} \sigma(y) dy = 0$, the quality composition remains the same over an interval $[\hat{x}_{i+1},\hat{x}_i]$ due to Lemma 4, then:

$$\rho(\hat{x}_{i}-)\left(E_{m}^{i+1}-v^{i+1}\right)=\rho(\hat{x}_{i+1})\left(E_{m}^{i+1}-v^{i+1}\right)$$

Using Equation (17) from Step 1, we have:

$$\rho(\hat{x}_{i+1}-)\left(E_m^{i+2}-v^{i+1}\right) = \rho(\hat{x}_i-)\left(E_m^{i+1}-v^{i+1}\right) < \rho(\hat{x}_i-)\left(E_m^{i+1}-v^i\right) \le CS$$

We obtain a contradiction with the hypothesis of the induction step: $CS = \rho(\hat{x}_{i+1}-) (E_m^{i+2} - v^{i+1})$. Hence, some locations at prices between v^i and v^{i+1} are visited: $\int_{y \in [\hat{x}_{i+1}, \hat{x}_i)}^{\hat{x}_i} \sigma(y) dy > 0$. Similar to the proof in the initial iterative step, consumers visit locations that have a payoff converging to $\rho(\hat{x}_i-)(E_m^{i+1}-v^i)$ from below. This completes the proof by induction. **Step 2**. We now combine the two steps. As a_m is the lowest-always-purchased quality, then $\mathbf{q}(l|\hat{x}_{a_m}) = \mathbf{q}(l|\hat{x}_{a_m}), \forall l$. Conditional on purchase on the interval $(0, \hat{x}_{a_m-1})$, consumer receives payoff:

$$E_m^{a_m} = \frac{\sum_{l \ge a_m} \pi(l) v^l}{\sum_{l \ge a_m} \pi(l)}$$

Using Step 1, the purchase probability at \hat{x}_{a_m-1} satisfies:

$$CS = \rho_m(\hat{x}_{a_m-1}-)\left(E_m^{a_m} - v^{a_m-1}\right) = \rho(\hat{x}_{a_m-1}-)\left(\frac{\sum_{l \ge a_m} \pi(l)v^l}{\sum_{l \ge a_m} \pi(l)} - v^{a_m-1}\right),$$

which implies that $\rho(\hat{x}_{a_m-1}-)$ is as in the statement of the lemma.

For every $i \in \{a_m - 1, e_m\}$, the product qualities are purchased with a positive probability, then $\hat{x}_i < 1$, and by Step 1:

$$CS = \rho_m(\hat{x}_{i-1}) \left(E_m^i - v^{i-1} \right) = \rho_m(\hat{x}_i) \left(E_m^{i+1} - v^i \right).$$

Using Equation (17), we can replace $\rho_m(\hat{x}_i) (E_m^{i+1} - v^i)$ with $\rho_m(\hat{x}_i) (E_m^i - v^i)$ in the above to obtain:

$$\frac{CS}{\rho(\hat{x}_i-)} = \frac{CS}{\rho(\hat{x}_i)} + v^i - v^{i-1}.$$

Lemma 19. For any sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ with positive total sales and consumer surplus $CS \ge 0$, if consumers purchase all quality types with positive probability, *i.e.* $e_m = 1$, then:

$$\frac{(1-\rho_m(\hat{x}_{e_m}-))}{\sum_{j\leq e_m}\pi(j)}\sum_{i=e^m}^{a_m-1}\left\{\left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}\frac{1-\rho_m(\hat{x}_i-)}{\rho_m(\hat{x}_i-)}\right)+\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}-\frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)}\right]\sum_{l\leq i}\pi(l)\right\}$$

$$=\frac{D(\hat{x}_{e_m})}{1-D(\hat{x}_{e_m})+\gamma},$$

and if $e_m > 1$, then

$$\frac{(1-\rho_m(\hat{x}_{e_m-1}))}{\sum_{j\leq e_m-1}\pi(j)}\sum_{i=e^m-1}^{a_m-1}\left\{\left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}\frac{1-\rho_M(\hat{x}_i-)}{\rho_m(\hat{x}_i-)}\right)+\frac{\rho(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}-\frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)}\right]\sum_{l\leq i}\pi(l)\right\}$$

Proof. By Lemma 4, $(1 - \rho(x))S_m(x)$ remains constant over any $(\hat{x}_{i+1}, \hat{x}_i)$, in addition since $\partial_x S_m(x) = -\rho_m(x)\sigma(x)$, and we obtain that over $(\hat{x}_{i+1}, \hat{x}_i)$:

$$\partial_x \rho(x) = -\rho_m(x)(1 - \rho_m(x))\frac{\sigma(x)}{S_m(x)}.$$
(18)

Recall that e_m is the lowest quality that is purchased with positive probability, then we have $\sum_{l \leq e_m} \mathbf{q}(l|x) S_m(x)$ remains constant over $[0, \hat{x}_{e_m})$. In addition, Lemma 4 implies there is no relative sorting between any qualities that are not purchased over $(0, \hat{x}_i)$. In particular, the relative quality composition between qualities below *i* and qualities below e_m stays constant, and for every $x \in (\hat{x}_{i+1}, \hat{x}_i)$:

$$(1 - \rho_m(x)) = \sum_{l \le e_m} \mathbf{q}(l|x) = \sum_{l \le e_m} \mathbf{q}(l|x) \frac{\sum_{l \le i} \pi(l)}{\sum_{j \le e_m} \pi(j)}.$$

This lets us rewrite Equation (18) as:

$$\partial_x \rho_m(x) = -\rho_m(x)(1-\rho_m(x))^2 \frac{\sigma(x)}{(1-\rho(\hat{x}_{e_m}-))S_m(\hat{x}_{e_m})} \frac{\sum_{j \le e_m} \pi(j)}{\sum_{l \le i} \pi(l)} \frac{\sum_{l \le i} \pi(l)}{\rho_m(x)(1-\rho_m(x))^2} = \frac{\sigma(x)}{(1-\rho_m(\hat{x}_{e_m}-))S_m(\hat{x}_{e_m})}$$

Integrating both sides of the above over $x \in (\hat{x}_{i+1}, \hat{x}_i)$, we get:

$$\frac{\sum_{l \le i} \pi(l)}{\sum_{j \le e_m} \pi(j)} \left[\ln \left(\frac{\rho_m(\hat{x}_{i+1})}{1 - \rho(\hat{x}_{i+1})} \frac{1 - \rho_m(\hat{x}_i)}{\rho(\hat{x}_{i-1})} \right) + \frac{\rho_m(\hat{x}_{i+1})}{1 - \rho_m(\hat{x}_{i+1})} - \frac{\rho_m(\hat{x}_i)}{1 - \rho_m(\hat{x}_{i-1})} \right] \\ = \frac{D(\hat{x}_i) - D(\hat{x}_{i+1})}{(1 - \rho_m(\hat{x}_{e_m}))S_m(\hat{x}_{e_m})}$$

Summing up the above over all intervals $[\hat{x}_{a_m}, \hat{x}_{a_m} - 1), \ldots, (\hat{x}_{e_m+1}, \hat{x}_{e_m}]$, we get:

$$\frac{(1-\rho_m(\hat{x}_{e_m}-))}{\sum_{j\leq e_m}\pi(j)}\sum_{i=e^m}^{a_m-1}\left\{\left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}\frac{1-\rho_M(\hat{x}_i-)}{\rho_m(\hat{x}_i-)}\right)+\frac{\rho(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}-\frac{\rho_m(\hat{x}_i-)}{1-\rho_m(\hat{x}_i-)}\right]\sum_{l\leq i}\pi(l)\right\}$$
$$=\frac{D(\hat{x}_{e_m})}{S_m(\hat{x}_{e_m})}=\frac{D(\hat{x}_{e_m})}{1-D(\hat{x}_{e_m})+\gamma}$$

When $e_m > 1$, then we can similarly sum over $[\hat{x}_{a_m}, \hat{x}_{a_m} - 1), \dots, (\hat{x}_{e_m}, \hat{x}_{e_m-1}]$ to get:

$$\frac{(1-\rho_m(\hat{x}_{e_m-1}))}{\sum_{j\leq e_m-1}\pi(j)}\sum_{i=e^m-1}^{a_m-1}\left\{\left[\ln\left(\frac{\rho_m(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}\frac{1-\rho_M(\hat{x}_{i}-)}{\rho_m(\hat{x}_{i}-)}\right)+\frac{\rho(\hat{x}_{i+1})}{1-\rho_m(\hat{x}_{i+1})}-\frac{\rho_m(\hat{x}_{i}-)}{1-\rho_m(\hat{x}_{i}-)}\right]\sum_{l\leq i}\pi(l)\right\}$$
$$=\frac{D(\hat{x}_{e_m-1})}{S_m(\hat{x}_{e_m-1})}=\frac{1}{\gamma}$$

Finally, we derive a general formulation of the irrelevance result for the total surplus.

Lemma 20. Consider sorting equilibrium $m = (\mathbf{p}, \sigma, \mathbf{q}, \gamma) \in \mathcal{E}$ with positive total sales and consumer surplus $CS \ge 0$. If consumers purchase all quality types with positive probability, *i.e.* $e_m = 1$, the total surplus is:

$$TS(m) = S_m(\hat{x}_{e_m}) \left(1 - \rho_m(\hat{x}_{e_m} -)\right) \left[\sum_{i=a_m}^{e_m+1} E_m^i \left(\frac{1}{1 - \rho_m(\hat{x}_i)} \frac{\sum_{j \le e_m-1} \pi(j)}{\sum_{j \le i} \pi(i)} - 1\right) + E_m^{e_m}\right] - E_m^{e_m} \gamma$$

And if some qualities are only cleared through disposal i.e. $e_m > 1$, the total surplus is:

$$TS(m) = \gamma \left(1 - \rho_m(\hat{x}_{e_m-1})\right) \left[\sum_{i=a_m}^{e_m+1} E_m^i \left(\frac{1}{1 - \rho_m(\hat{x}_i)} \frac{\sum_{j \le e_m} \pi(j)}{\sum_{j \le i} \pi(i)} - 1\right) + E_m^{e_m}\right] - E_m^{e_m} \gamma$$

Proof. Case 1: $e_m = 1$. The total surplus is:

$$TS(m) = \sum_{i=a_m}^{e_m+1} E_m^i \left(S_m(\hat{x}_i) - S_m(\hat{x}_{i-1}) \right) + E_m^{e_m} \left(S_m(\hat{x}_{e_m}) - \gamma \right).$$

By Lemma 4, for every $i \in \{e_m + 1, a_m\}$: $(1 - \rho(\hat{x}_i))S(\hat{x}_i) = (1 - \rho(\hat{x}_{i-1}))S(\hat{x}_{i-1})$. Then, we can replace $S(\hat{x}_i)$ by:

$$S(\hat{x}_i) = \frac{1 - \rho_m(\hat{x}_{i-1})}{(1 - \rho_m(\hat{x}_i))} S(\hat{x}_{i-1}) = S(\hat{x}_{e_m}) \prod_{j=i}^{e^m - 1} \frac{1 - \rho_m(\hat{x}_{i+1})}{(1 - \rho_m(\hat{x}_j))}$$

$$= S(\hat{x}_{e_m}) \frac{1 - \rho_m(\hat{x}_{e_m})}{1 - \rho_m(\hat{x}_i)} \prod_{j=i+1}^{e^m - 1} \frac{1 - \rho_m(\hat{x}_j)}{(1 - \rho_m(\hat{x}_j))}$$

By Lemma 17, for each i the jump in the purchase probability satisfies:

$$\frac{1 - \rho_m(\hat{x}_j)}{(1 - \rho_m(\hat{x}_j))} = \frac{\sum_{l \le j} \pi(j)}{\sum_{l \le j} \pi(j)},$$

which implies from the above:

$$S(\hat{x}_i) = S(\hat{x}_{e_m}) \frac{1 - \rho_m(\hat{x}_{e_m})}{1 - \rho_m(\hat{x}_i)} \frac{\sum_{j \le e_m - 1} \pi(j)}{\sum_{j \le i} \pi(i)}.$$

$$TS(m) = \sum_{i=a_m}^{e_m+1} E_m^i \left(S_m(\hat{x}_i) - S_m(\hat{x}_{i-1}) \right) + E_{e_m} \left(S_m(\hat{x}_{e_m}) - \gamma \right)$$
$$= S_m(\hat{x}_{e_m}) \left(1 - \rho_m(\hat{x}_{e_m-1}) \right) \left[\sum_{i=a_m}^{e_m+1} E_m^i \left(\frac{1}{1 - \rho_m(\hat{x}_i)} \frac{\sum_{j \le e_m-1} \pi(j)}{\sum_{j \le i} \pi(i)} - 1 \right) + E_m^{e_m} \right] - E_m^{e_m} \gamma$$

Case 2: $e_m > 1$ is derived analogously. The only difference is that there is no jump at $\hat{x}_{e_m-1} = 1$.

Taking stock, Lemma 18 implies that the purchase probabilities at the threshold locations are the same at the two sorting equilibria m_1 , m_2 if they have the same consumer payoff CS. By Lemma 17, the conditional expected value of the products is similarly the same. If $e_{m_1} = e_{m_2} = 1$, by Lemma 19, both sorting equilibria have the same downstream sales at $\hat{x}_{e_{m_1}}$ and $\hat{x}_{e_{m_2}}$, respectively. Then, by Lemma 20, the two sorting equilibria have the same total surplus.

If $e_{m_1} = e_{m_2} > 1$, by Lemma 19, then both equilibria have the same downstream sales at $\hat{x}_{e_{m_1}-1} = \hat{x}_{e_{m_2}-1} = 1$. Both equilibria then deliver the same total surplus.

By assumption, they induce the same consumer payoff. As the total surplus is the same across the two sorting equilibria, the seller's payoff must be the same. This concludes the proof.